Find a feasible point for \( Ax \leq b \). \( A \) is \( mn \).

Have an ellipsoid \( E_k \) that contains \( \text{Ex: } Ax \leq b \).

Know center \( z_k \) of \( E_k \).

If \( Ax \leq b \), done.

Else \( z_k \) violates one of the constraints, say \( a_i z_k \geq b_i \)
where \( a_i^T \) is the row of \( A \).

Then \( \text{Ex: } Ax \leq b \) \( \subseteq E_k \cap \{ x \mid a_i^T x \leq b_i \} \).

So construct a new ellipsoid \( E_{k+1} \geq E_k \cap \{ x \mid a_i^T x \leq b_i \} \),
with new center \( z_{k+1} \).
Def. A matrix \( M \) is positive definite if \( x^T M x > 0 \) whenever \( x \neq 0 \).

Def. \( M \) is positive definite if all the eigenvalues of \( M \) are positive.

Def. An elliptoid is a collection of points 

\[ \{ x \in \mathbb{R}^n : (x - \mu)^T M^{-1} (x - \mu) \leq 1 \} \]

where \( M \) is a symmetric positive definite matrix, \( \mu \) is the center of the ellipsoid.

\[ \text{Eq. (1)} \quad M = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}, \quad \mu = \begin{bmatrix} 3 \\ -2 \end{bmatrix}. \]

Then 

\[ (x - \mu)^T M^{-1} (x - \mu) \leq 1 \iff \frac{1}{4} (x_1 - 3)^2 + \frac{1}{9} (x_2 + 2)^2 \leq 1 \]
(2) \[ M = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \]  
\[ \text{null}(M) = \text{span}(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}). \]

\[ z = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \]

\[ S_0, \text{ an ellipse of } B \]

\[ x \mapsto \|x-z\|, M = B^T B^{-1} \]

\[ S_0, \text{ in order to describe a ellipsoid, we need to give the center } z \]

and the matrix \( M \).

Initially, we take \( E_0 = \{ x : x^T x \leq 1 \} \),

\[ z_0 = 0, M_0 = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \]

so \( E_0 \) is a bound on version of the unit ball. It suffices to take \( L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \).

\[ L = 2^{-\frac{1}{2}}, \text{ when } v = \sin \phi \in [0, \pi] \text{ for } \phi \in \mathbb{R}. \]

Given \( z_k, a_k \), get \( z_{k+1}, a_{k+1} \) as follows:

\[ z_{k+1} = z_k - \frac{1}{n+1} \frac{a_k}{\sqrt{a_k^T M_k a_k}} \]

\[ M_{k+1} = \frac{n}{n+1} \left( M_k - \frac{z_k}{n+1} \frac{a_k a_k^T M_k}{a_k^T M_k a_k} \right) \]
We have the following result:

\[
\frac{\text{Vol} \left( \mathbb{E}^{k+1} \right)}{\text{Vol} \left( \mathbb{E}^k \right)} < e^{-\frac{1}{16n^2}}.
\]

Let \( \mathcal{P} = \{ x : \|Ax\| \leq \delta \} \) be bounded and full-dimensional.

Then \( \text{Vol}(\mathcal{P}) \geq 2^{-2n\delta^2} \), \( \delta \) as before.

So, now assume either \( \mathcal{P} = \emptyset \) or \( \mathcal{P} \) is full-dimensional.

We have \( \mathbb{E}^k \supseteq \mathcal{P} \) for all \( k \).

Also, \( \text{Vol} \left( \mathbb{E}^{k+1} \right) < e^{-\frac{1}{16n^2}} \text{Vol} \left( \mathbb{E}^k \right) \)

\[
< \cdots < e^{-\frac{(k+1)}{16n^2}} \text{Vol} \left( \mathbb{E}^0 \right) 
\]

\[
\leq e^{-\frac{(k+1)}{16n^2}} (2\delta)^n.
\]

So, eventually we'll either have \( x^{(k)} \notin \mathcal{P} \) or we'll get \( \text{Vol}(\mathbb{E}^{k+1}) \leq 2^{-2n\delta} \), showing that \( \mathcal{P} \) is empty.

Let \( k = 16n^2 \nu \)

Then \( \text{Vol}(\mathbb{E}^k) \leq 2^{-2n\delta} \).

If \( \mathbb{E} = \{ x : (x-\mu)^T \Sigma^{-1} (x-\mu) \leq \beta \} \) then \( \text{Vol}(\mathbb{E}) \propto \sqrt{\det(\Sigma)} \).
Complete Alg:

0. Initialize: \( M_0 = \Delta I, \quad z_{\infty} = 0, \quad k = 0. \)

1. Check feasibility: Is \( z_{k+1} \in R \)? If yes, stop, successful termination.
   If no, find i with \( a_i^T z_k > b_i \).

2. Shrink ellipsoid: \( z_{k+1} = z_k - \frac{1}{n+1} \frac{M_k a_i}{\sqrt{a_i^T M_k a_i}} \)
   \( M_{k+1} = \frac{1}{2} M_k - \frac{2}{n+1} \frac{M_k a_i (a_i^T M_k a_i)}{a_i^T M_k a_i} \)
   \( k \leftarrow k + 1 \)

3. Ellipsoid too small?
   If \( k \geq 16\varepsilon^2 \sqrt{n} \), stop: \( P \) is empty.
   Else, return to 1.

Minimizing a linear function over an ellipsoid:

Consider the problem \( \min_x \quad d^T x \) \hspace{1cm} (d \not= 0)

subject to \( (x-z)^T M^{-1} (x-z) \leq 1 \)

What conditions for this problem:

\( d + 2uM^{-1}(x-z) = 0 \)

So optimal \( x \) is \( \bar{x} = z - \frac{1}{2u} M d \).

The constraint must hold at equality, so \( (x-z)^T M^{-1} (x-z) = 1 \)

Thus \( \frac{1}{4u} d^T S d = 1 \), so \( u = \sqrt{\frac{d^T S d}{4}} \) and \( \bar{x} = z - \frac{Md}{\sqrt{d^T S d}} \).
Hence the update

$$Z_{k+1} = Z_k - \frac{1}{\lambda_k+1} \frac{M_k}{\sqrt{\lambda_k+1}}$$

moves in the direction of minimizing $c^T x$, but not very close in that direction.

$$c^T x = c^T Z_k$$

Since $P$ may not be full dimensional, the $Q = \{ x : A x \leq b + \varepsilon \}$

If $\varepsilon$ is small enough, $P$ is empty $\iff Q$ is nonempty.

Solving $\min Q c^T x : A x \leq b$.

If $z_k$ satisfies $A x \leq b$, update:

$$Z_{k+1} = Z_k - \frac{1}{\lambda_k+1} \frac{M_k}{\sqrt{\lambda_k+1}}$$

$$M_{k+1} = \frac{n^2}{n^2 + 1} \left( \frac{M_k - \frac{2}{n+1} \frac{M_k c c^T M_k}{c^T M_k}}{c^T M_k} \right)$$
In practice, the worst case bound is not competitive with simplex. Even references like "deep cut" don't really help.

Deep cut:

\[ \frac{1}{2} a_i x = a_i \bar{z} \quad a_i x = b_i \]

Get any possible cut, so long that ellipsoid is repeatable. I think it of centroid.

Equivalence of separator & optimizer.