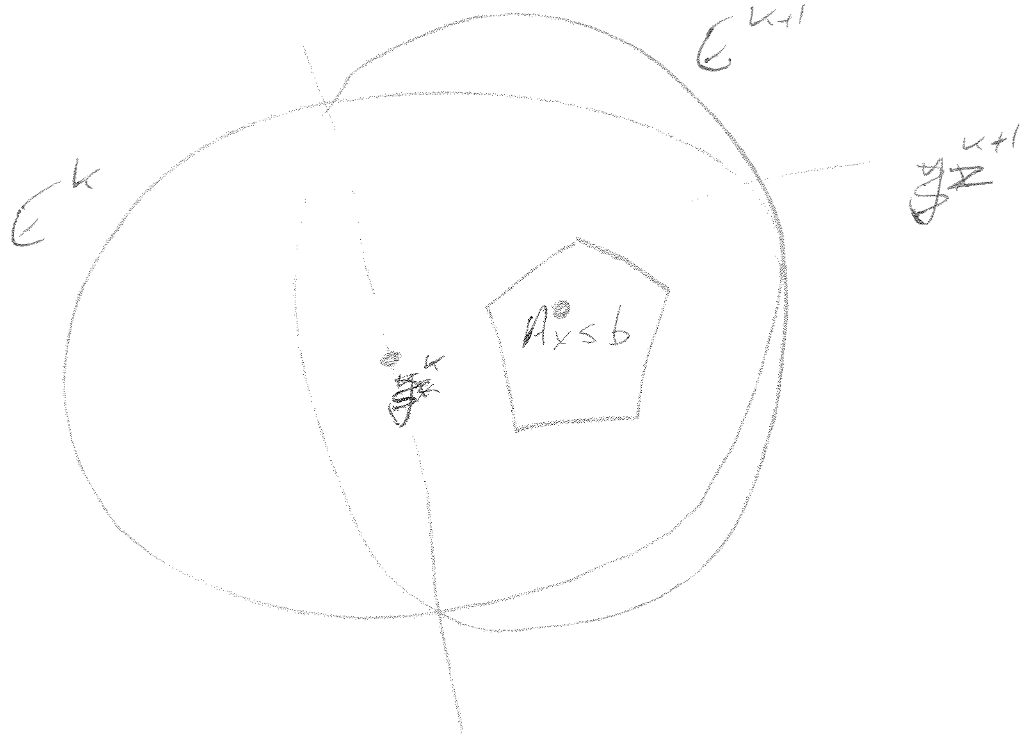
THE ELLIPSOID ALGORITHM FOR LINEAR PROGRAMMING.

(Khachian 1979, Shor, Nemirovski, Yudin 1970, 1976, 19[illegible])

Find a feasible point for $Ax \le b$ A is $m \times n$.



Have an ellipsoid E^k which contains $\{x : Ax \le b\}$.

Know center z^k of E^k.

If $Az^k \le b$, done

Else z^k violates one of the constraints, say $a_i^T z^k > b_i$

where a_i^T is ith row of A.

Then $\{x : Ax \le b\} \subseteq E^k \cap \{x : a_i^T x \le a_i^T z^k\}$

So construct a new ellipsoid $E^{k+1} \supseteq E^k \cap \{x : a_i^T x \le a_i^T z^k\}$, with new center z^{k+1}

Representing ellipsoids:

Def A matrix M is POSITIVE DEFINITE if $x^T M x > 0$ whenever $x \neq 0$

M is pos def iff all the eigenvalues of M are positive.

If M is pos def then M^{-1} is pos def (evals of M^{-1} are $\frac{1}{\text{evals}}$ of

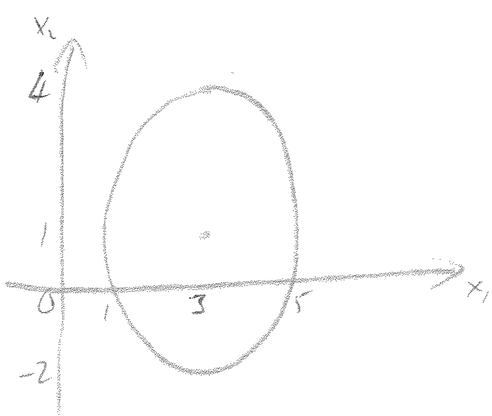
Def An ELLIPSOID is a collection of points

$$\text{ell}(z, M) = \{ x \in \mathbb{R}^n : (x-z)^T M^{-1} (x-z) \leq 1 \}$$

where M is a symmetric positive definite matrix. z is the CENTER of the ellipsoid.

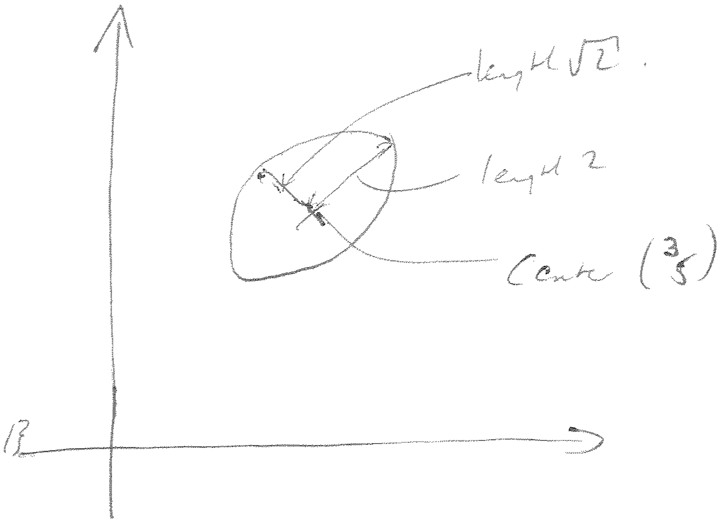
Eg (1) $M = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$ $z = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Then $(x-z)^T M^{-1} (x-z) \leq 1 \iff \frac{1}{4}(x_1 - 3)^2 + \frac{1}{9}(x_2 + 1)^2 \leq 1$



(2) $M = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ Evals 2, 4 evecs $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$z = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$



So, an ellipsoid is an affine transformation of the unit ball $S = \{x : x^T x \le 1\}$

$x \to B(x - z)$, $M = B^{-1} B^{-T}$.

So, in order to describe an ellipsoid, we need to give the center z and the matrix M.

Initially, we take $E_0 = \{x : \frac{1}{\Delta} x^T x \le 1\}$,

ie $z_0 = 0$, $M_0 = \Delta I$, so E_0 is a blown up version of the unit ball. It suffices to take

$\Delta = 2^{4n^2 v}$, where v = size of a row of $[A \;\; b]$.

Given M_k, z_k get M_{k+1}, z_{k+1} as follows:

$$z_{k+1} = z_k - \frac{1}{n+1} \frac{M_k a_i}{\sqrt{a_i^T M_k a_i}}, \text{ where } a_i^T z_k > b_i$$

$$M_{k+1} = \frac{n^2}{n^2 - 1} \left(M_k - \frac{2}{n+1} \frac{M_k a_i a_i^T M_k}{a_i^T M_k a_i} \right)$$

We have the following results:

Lemma $$\frac{\mathrm{Vol}(E^{k+1})}{\mathrm{Vol}(E^k)} < e^{-1/(2n+2)}.$$

Lemma ~~Also assume~~ Let $P = \{x \mid Ax \le b\}$ be bounded and full-dimensional.
Then $\mathrm{Vol}(P) \ge 2^{-2nv}$, v as before.

So: Now assume either $P = \phi$ or P is full-dimensional.

We have $E^k \supseteq P$ for all k.

$$\begin{aligned} \text{Also, } \mathrm{Vol}(E^{k+1}) &< e^{-1/(2n+2)}\,\mathrm{Vol}(E^k) \\ &< \cdots \\ &< e^{-(k+1)/(2n+2)}\,\mathrm{Vol}(E^0) \\ &\le e^{-(k+1)/(2n+2)}\,(2\Delta)^n. \end{aligned}$$

So eventually we'll either have x^{k+1} in P or we'll get $\mathrm{Vol}(E^{k+1}) < 2^{-2nv}$, showing that P is empty.

Let $K = 16n^2v$

Then $\mathrm{Vol}(E^K) < 2^{-2nv}$.

If $E = \{x : (x-z)^T M^{-1}(x-z) \le B\}$

then $\mathrm{vol}(E) \propto \sqrt{\det(M)}$.

Complete Algo:

0. Initialize: $M_0 = \Delta I$, $z_0 = 0$, $k = 0$.

1. Check feasibility: Is z_k in P? If YES, stop, successful termination.
   If NO, find i with $a_i^T z_k > b_i$.

2. Shrink ellipsoid: $$z_{k+1} = z_k - \frac{1}{n+1} \frac{M_k a_i}{\sqrt{a_i^T M_k a_i}}$$

$$M_{k+1} = \frac{n^2}{n^2-1}\left(M_k - \frac{2}{n+1} \frac{M_k a_i a_i^T M_k}{a_i^T M_k a_i}\right)$$

$$k \leftarrow k+1$$

3. Ellipsoid too small?

   If $k \geq 16n^2 v$, STOP: P is empty.

   Else, return to 1.

Minimizing a linear function over an ellipsoid:

Consider the problem $$\min_x \; d^T x \qquad (d \neq 0).$$
$$\text{s.t. } (x-z)^T M^{-1} (x-z) \leq 1$$

KKT conditions for this problem:

$$d + 2u M^{-1}(x-z) = 0.$$

So optimal x is $\bar{x} = z - \frac{1}{2u} M d$.

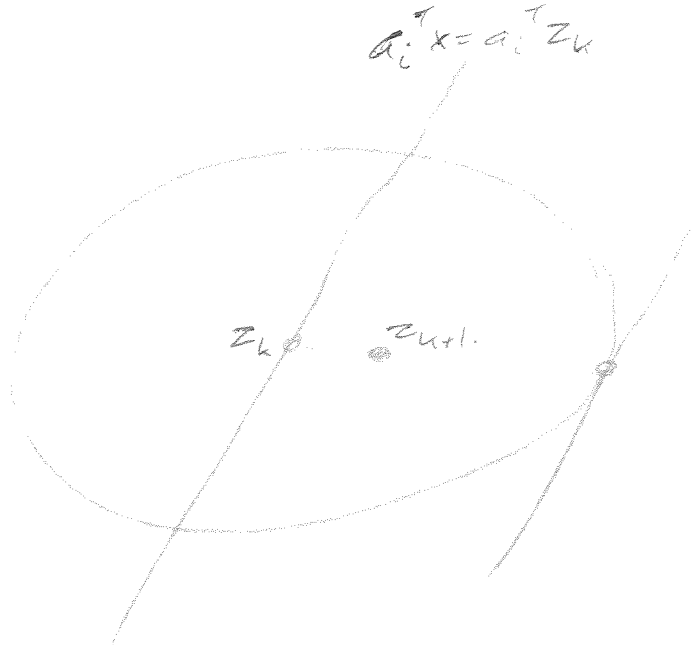
The constraint must hold at equality, so $(x-z)^T M^{-1}(x-z) = 1$

Thus $\frac{1}{4u^2} d^T M d = 1$, so $u = \frac{\sqrt{d^T M d}}{2}$ and $\bar{x} = z - \frac{Md}{\sqrt{d^T M d}}$.

Hence the update

$$z_{k+1} = z_k - \frac{1}{n+1}\frac{M a_i}{\sqrt{a_i^T M a_i}}$$

moves in the direction of minimizing $a_i^T x$, but not very far in that direction.



Notes

- Since P may not be full dimensional, take $Q = \{x : Ax \leq b + \varepsilon\}$

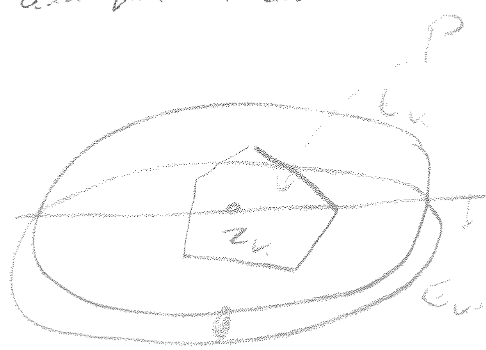
  If ε is small enough, P is nonempty $\Leftrightarrow$ Q is nonempty and full-dimensional.

- Solving $\min\{c^T x : Ax \leq b\}$.

  If z_k satisfies $Ax \leq b$, update:

$$z_{k+1} = z_k - \frac{1}{n+1}\frac{Mc}{\sqrt{c^T M c}}$$

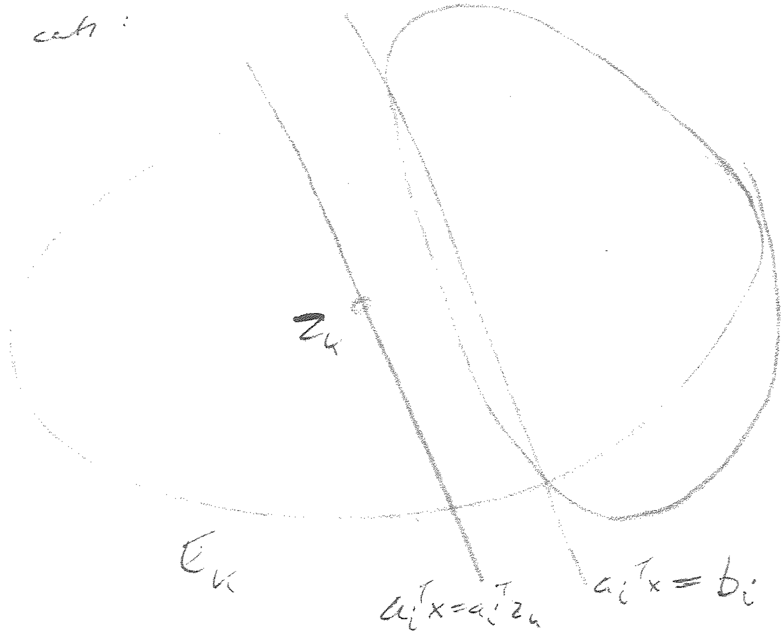
$$M_{k+1} = \frac{n^2}{n^2-1}\left(M_k - \frac{2}{n+1}\frac{M_k c c^T M_k}{c^T M c}\right)$$



In practice

Behaves like worst case bound, so not competitive with simplex. Even refinements like "deep cuts" don't really help:

Deep cuts:



Get many parallel cuts, so long thin ellipsoids,

~~Equival~~

Infinite # of constraints

Equivalence of separation & optimization.