Revised Simplex Method

Simplex algorithm, but implemented more efficiently.

Example:

$$\min \quad -x_1 - 2x_2 - x_3 \qquad + 3x_5$$

$$\begin{aligned} \text{st.} \quad x_1 - 2x_2 \qquad\quad + x_4 \qquad\qquad &= 1 \\ 3x_2 - x_3 + 2x_4 + x_5 &= 1 \\ 2x_1 \qquad + x_3 \qquad\quad - 2x_5 &= 8 \\ x_i \geq 0. \end{aligned}$$

Initial bfs: x_1, x_2, x_3 basic $\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & -1 \\ 2 & 0 & 1 \end{bmatrix}$

Value: -7.

FIRST ITERATION:

Reduced costs: $c_N^T - c_B^T B^{-1} N$.

First find $y^T = c_B^T B^{-1}$, so $y^T B = c_B^T$, so $B^T y = c_B$.

So solve this system of equations:

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ -2 & 3 & 0 & -2 \\ 0 & -1 & 1 & -1 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 3 & 4 & -4 \\ 0 & -1 & 1 & -1 \end{bmatrix} \xrightarrow{R_3 + \frac{1}{3}R_2} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 3 & 4 & -4 \\ 0 & 0 & \frac{7}{3} & -\frac{7}{3} \end{bmatrix}$$

So $\left.\begin{aligned} y_1 \qquad + 2y_3 &= -1 \\ 3y_2 + 4y_3 &= -4 \\ \tfrac{7}{3} y_3 &= -\tfrac{7}{3} \end{aligned}\right\} \Rightarrow y_3 = -1, y_2 = 0, y_1 = +1$. ie $y = \begin{bmatrix} +1 \\ 0 \\ -1 \end{bmatrix}$

Now $c_N^T - c_B^T B^{-1} N = \begin{bmatrix} 0 & 3 \end{bmatrix} - \begin{bmatrix} +1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & +2 \end{bmatrix}$

$$= \begin{array}{cc} [-1 & 1]. \\ x_4 & x_5. \end{array}$$

Thus, choose x_4 to enter basis.

Need to calculate the appropriate column of $B^{-1}N$,

ie $d = B^{-1}\begin{bmatrix}1\\2\\0\end{bmatrix}$. So solve $Bd = \begin{bmatrix}1\\2\\0\end{bmatrix}$

$$\begin{bmatrix}1 & -2 & 0 & 1\\0 & 3 & -1 & 2\\2 & 0 & 1 & 0\end{bmatrix} \xrightarrow{R_3-2R_1} \begin{bmatrix}1 & -2 & 0 & 1\\0 & 3 & -1 & 2\\0 & 4 & 1 & -2\end{bmatrix} \xrightarrow[R_3-\frac{4}{3}R_2]{} \begin{bmatrix}1 & -2 & 0 & 1\\0 & 3 & -1 & 2\\0 & 0 & \frac{7}{3} & -\frac{14}{3}\end{bmatrix}$$

So
$$\left.\begin{aligned} d_1 - 2d_2 \quad &= 1\\ 3d_2 - d_3 &= 2\\ \tfrac{7}{3}d_3 &= -\tfrac{14}{3}\end{aligned}\right\} \Rightarrow d_3 = -2,\ d_2 = 0,\ d_1 = 1 \quad \text{ie } d = \begin{bmatrix}1\\0\\-2\end{bmatrix}$$

Minimum ratio test:

Find $\Delta = \min\left\{ \frac{(B^{-1}b)_i}{d_i} : d_i > 0 \right\}$

Now $B^{-1}b = \text{current bfs} = \begin{bmatrix}3\\1\\2\end{bmatrix}$.

So $\Delta = \min\left\{ \frac{3}{1}, \cancel{\left(\frac{1}{0}\right)}, \cancel{\left(\frac{2}{-2}\right)} \right\} = 3$, achieved by first row.

So x_4 enters at value 3, basic variable in first row (ie x_1) leaves the basis.

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}\bar{x}_1\\\bar{x}_2\\\bar{x}_3\end{bmatrix} - \Delta d = \begin{bmatrix}3\\1\\2\end{bmatrix} - 3\begin{bmatrix}1\\0\\-2\end{bmatrix} = \begin{bmatrix}0\\1\\8\end{bmatrix}.$$

So get new bfs: $\begin{bmatrix}\bar{x}_4\\\bar{x}_2\\\bar{x}_3\end{bmatrix} = \begin{bmatrix}3\\1\\8\end{bmatrix}$ $\quad B = \begin{bmatrix}1 & -2 & 0\\2 & 3 & -1\\0 & 0 & 1\end{bmatrix}$.

SECOND ITERATION:

Reduced costs: $c_N^T - c_B^T B^{-1} N$

First find y which solves $B^T y = c_B$

So solve:

$$\begin{bmatrix} 1 & +2 & 0 & 0 \\ -2 & 3 & 0 & -2 \\ 0 & -1 & 1 & -1 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & +2 & 0 & 0 \\ 0 & 7 & 0 & -2 \\ 0 & -1 & 1 & -1 \end{bmatrix} \xrightarrow{R_3+\frac{1}{7}R_2} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & -\frac{9}{7} \end{bmatrix}$$

So
$$\left.\begin{aligned} y_1 + 2y_2 &= 0 \\ 7y_2 &= -2 \\ y_3 &= -9/7 \end{aligned}\right\} \quad y_3 = -9/7,\ y_2 = \frac{-2}{7},\ y_1 = \frac{+4}{7}, \text{ ie } y = \begin{bmatrix} +4/7 \\ -2/7 \\ -9/7 \end{bmatrix}.$$

Now ~~red~~ $c_N^T - c_B^T B^{-1} N = c_N^T - y^T N = [-1 \quad 3] - [\tfrac{+4}{7} \quad \tfrac{-2}{7} \quad -\tfrac{9}{7}] \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -2 \end{bmatrix}$

$= [-1 \quad 3]$ ~~$-[\ \ ]$~~ $- [-2 \quad \tfrac{16}{7}]$

~~$= [\tfrac{5}{7} \quad \tfrac{16}{7}]$~~ > 0, so optimal.

$= [1 \quad \tfrac{5}{7}] > 0$, so optimal.

Notes:

(1) $d = B^{-1} a^k$ = direction in ~~bas~~ current basic variables.

(2) Dual problem is
$$\max\ b^T y$$
$$B^T y \le c_B$$
$$N^T y \le c_N$$

So if we choose $y = B^{-T} c_B$, then y is dual feasible provided $N^T y \le c_N$.

ie, provided $0 \le c_N - N^T y = c_N - N^T B^{-T} c_B$

ie, provided all reduced costs are nonnegative.

Further dual value $= b^T B^{-T} c_B = c_B^T B^{-1} b$ = primal value.

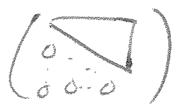
So if red costs ≥ 0, then y is dual optimal.

Improving the linear algebra

We need to solve $B^T y = c_B$ and $Bd = a^k$.

Assume we know L, U such that $LB = U$,

where L is lower triangular ,

U is upper triangular .

~~Then solving $B^T y = c_B$~~

Notice that it is easy to solve systems of the form $Lp = q$

(p unknown, q known) Eg:

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 1 & 1 \end{bmatrix} \quad q = \begin{bmatrix} 4 \\ 5 \\ -4 \end{bmatrix} \quad p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

So
$$\left.\begin{aligned} 2p_1 \qquad\qquad &= 4 \\ p_1 + 3p_2 \qquad &= 5 \\ -p_1 + p_2 + p_3 &= -4 \end{aligned}\right\} \; p_1 = 2,\; p_2 = 1,\; p_3 = -3$$

Similarly, easy to solve systems of the form $Up = q$ (p unknown, q known).

Then $B^T y = c_B$ is equivalent to solving

$U^T L^{-T} y = c_B$, which is equivalent to

Solve $U^T z = c_B$, then calculate $y = L^T z$.

And solving $Bd = a^k$ is equivalent to solving

$L^{-1} U d = a^k$, which is equivalent to

Calculate $q = La^k$ then solve $Ud = q$.

Thus, once we find a factorization of B such that $LB = U$, then solving $B^T y = c_B$, and $Bd = a^k$ are easy.

Such a factorization can be found by Gaussian elimination:

Let $B = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & -1 \\ 2 & 0 & 1 \end{bmatrix}$, as in the example.

If we were running Gaussian elimination, we'd subtract $2(\text{row } 1)$ from row 3. In matrix terms, this is equivalent to premultiplying B by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ (Do unto the identity as you would do unto B.)

Then $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & -1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & -1 \\ 0 & 4 & 1 \end{bmatrix}$

Now $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{4}{3} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & -1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & \frac{7}{3} \end{bmatrix}$.

ie $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -\frac{4}{3} & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & -1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & \frac{7}{3} \end{bmatrix}$

ie $\qquad L \qquad B \qquad = \qquad U$.

In example, $c_B = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$

Solve $U^T z = c_B$: $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 0 & -1 & \frac{7}{3} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} \Rightarrow z = \begin{bmatrix} -1 \\ -\frac{4}{3} \\ -1 \end{bmatrix}$

Then $y = L^T z = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -\frac{4}{3} \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, as before.

Also, $a^4 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. So to find d with $Bd = a^4$,

Find $q = La^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -\frac{4}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -\frac{14}{3} \end{bmatrix}$

Solve $Ud = q$: $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & \frac{7}{3} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -\frac{14}{3} \end{bmatrix} \Rightarrow d = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ as before

What about the second iteration, so have new basis matrix

$\bar{B} = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix}$? Do we need to find a new $\bar{L}, \bar{U}$ from scratch, or can we use L, U?

B and $\bar{B}$ only differ in the first column. In fact

$$\bar{B} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & -1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \cancel{\bar{B}}\, B \left[d \;\middle|\; \begin{matrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix} \right],$$

i.e., BE_1, where E_1 only differs from the identity I in the first column, which is replaced by the vector d.

Thus $\bar{B} = B E_1$

so $L\bar{B} = LB\cancel{B}E_1 = UE_1$

Need to solve $\bar{B}^T y = c_{\bar{B}}$, $\bar{B} d = a^k$

Solving $\bar{B}^T y = c_{\bar{B}}$ is equivalent to ~~solving~~

Solving $E_1^T U^T L^{-T} y = c_{\bar{B}}$, ie,

Solving $E_1^T w = c_{\bar{B}}$, then solving $U^T z = w$, then calculating $y = L^T z$

So, in example: w solves

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} \Rightarrow w = \begin{bmatrix} -2 \\ -2 \\ -1 \end{bmatrix}$$

Then z solves

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 0 & -1 & \frac{7}{3} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -1 \end{bmatrix} \Rightarrow z = \begin{bmatrix} -2 \\ -2 \\ -\frac{9}{7} \end{bmatrix}$$

Then $$y = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \\ -\frac{9}{7} \end{bmatrix} = \begin{bmatrix} \frac{4}{7} \\ -\frac{2}{7} \\ -\frac{9}{7} \end{bmatrix}$$, as before.

Also, solving $\bar{B} d = a^k$ is equivalent to

Solving $L^{-1} U E_1 d = a^k$, ie

Calculating $q = La^k$, then solving $Up = q$, then solving $E_1 d = p$.

After several iterations: (Superscripts are ITERATION COUNTERS.)

Let B^0 be initial basis matrix. Know L, U so that $LB^0 = U$

After r iterations, basis matrix is $B^r = B^0 E^1 \dots E^r$

where each E^i is an ETA MATRIX, ie it differs from the identity in exactly one column. (ETA FACTORIZATION OF THE INVERSE, very similar to "PRODUCT FORM OF THE INVERSE", DANTZIG & ORCHARD-HAYS, 1954.)

Then can solve $B^{r^T} y = c_{B^r}$ by:

(i) Solve $E^{r^T} w^r = c_{B^r}$, then

solve $E^{r-1^T} w^{r-1} = w^r$, then

$\vdots$

solve $E^{1^T} w^1 = w^2$, then

(ii) Solve $U^T z = w$, then

(iii) Calculate $y = L^T z$

Can solve $B^r d = a^k$ by:

(i) Calculate $q = L a^k$, then

(ii) Solve $U p^0 = q$, then

(iii) Solve $E^1 p^1 = p^0$, then

Solve $E^2 p^2 = p^1$, then

$\vdots$

Solve $E^{r-1} p^{r-1} = p^{r-2}$, then

Solve $E^r d = p^{r-1}$.

Eventually should refactor basis matrix $\bar{L} B^r = \bar{U}$.

Resolution:

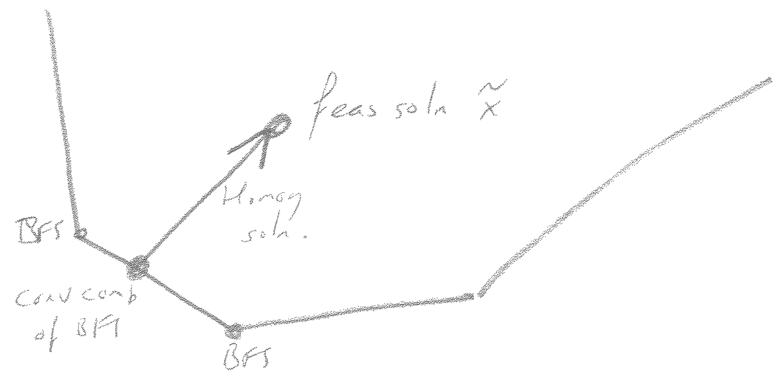
~~p.38~~: p.16: State theorem, give sketches contained in proof.
p.29: First half page (no cones, no polars).
Handout of F-M example.
p.38: Weyl's Thm, with proof.
p.29: Convex cone recall. Notes (1)-(4).
p.30: Definitions of conical dual, dual (polar) cone.
p.38: Cor 1 (No proof)
p.39: Minkowski's Thm (no proof)
p.40: Affine Weyl, with proof.
p.41: Affine Minkowski, no proof.
p.42-43: Everything after proof of affine Minkowski.

Resolution Theorem

Every feasible solution of (P) can be expressed as the sum of

(i) a convex combination of the BFS;

and (ii) a homogeneous solution corresponding to (P).



Proof Suppose $\tilde{x}$ is a feasible solution of (P).

Let J be the support of $\tilde{x}$, so $J = \{j : \tilde{x}_j > 0\} = \{j_1, \dots, j_k\}$, say.

So $\tilde{x}$ uses columns $A_{\cdot j_1}, \dots, A_{\cdot j_k}$ of A.

We use induction on k.

Base case: $k = 0$
Then $\tilde{x} = 0$, so $b = 0$. Thus $\tilde{x}$ is itself a BFS and a homog soln,
so $\tilde{x} = 0 + 0$.

Induction step: $k \geq 1$.

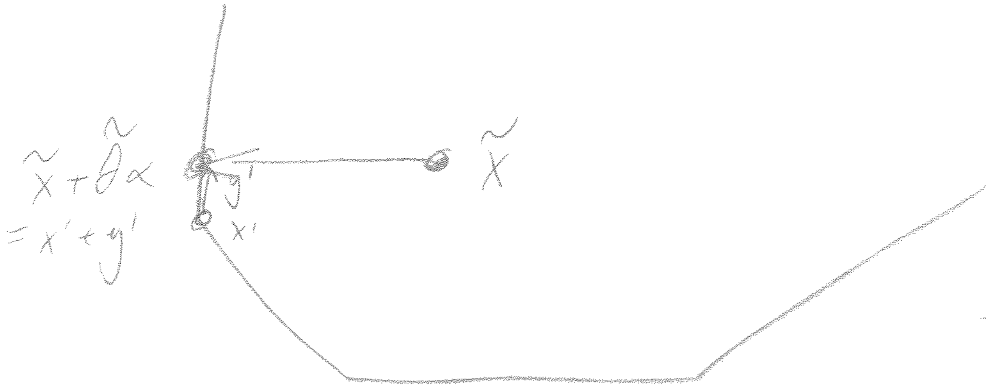
Suppose the result holds for every feasible solution that uses a set of $k-1$ or less column vectors of A.

If $\{A_{\cdot j_1}, \dots, A_{\cdot j_k}\}$ is lin indep, then $\tilde{x}$ is a BFS and the result holds.

So assume $\{A_{\cdot j_1}, \dots, A_{\cdot j_k}\}$ are lin. dep. So can find $\alpha_1, \dots, \alpha_k$ not all zero

with $\alpha_1 A_{\cdot j_1} + \dots + \alpha_k A_{\cdot j_k} = 0$

Can assume WLOG some component of α is < 0



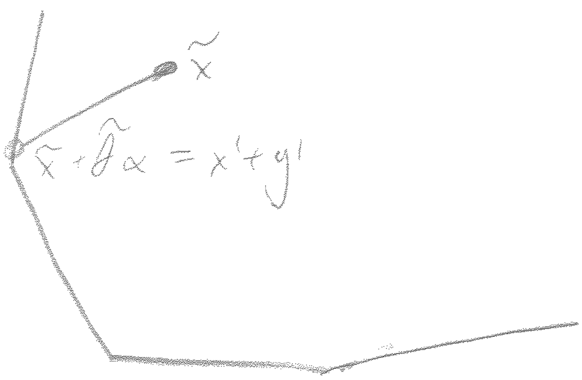
$\tilde{x} + \theta\alpha$ feasible for small enough $\theta > 0$.

Let $\tilde{\theta} = \min_{j \in J} \left\{ \frac{\tilde{x}_j}{-\alpha_j} : \alpha_j < 0 \right\} > 0$

Then $\tilde{x} + \tilde{\theta}\alpha$ uses at most $k-1$ columns of A.
So $\tilde{x} + \tilde{\theta}\alpha = x' + y'$ where x' is a conv comb of BFS and y' is homogeneous solution.

Two cases

(a) $\alpha \leq 0$

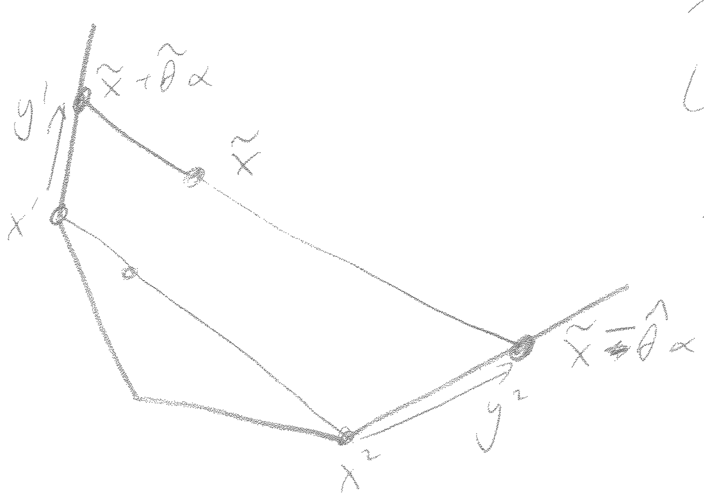


Rearrange: $\tilde{x} = x' + (y' - \tilde{\theta}\alpha)$

Now $y' - \tilde{\theta}\alpha \geq 0$ and $A(y' - \tilde{\theta}\alpha) = 0$,

so $y' - \tilde{\theta}\alpha$ is also homogeneous solution.

(b) α has a positive component



$\tilde{x} - \theta\alpha$ feasible for small enough $\theta > 0$

Let $\hat{\theta} = \min_{j \in J} \left\{ \frac{\tilde{x}_j}{\alpha_j} : \alpha_j > 0 \right\} > 0$

Then $\tilde{x} - \hat{\theta}\alpha$ uses at most $k-1$ columns of A
So $\tilde{x} - \hat{\theta}\alpha = x^2 + y^2$ where x^2 is conv comb of BFS and y^2 is homogeneous solution.

So we have $\tilde{x} + \tilde{\theta}\alpha = x^1 + y^1$ and $\tilde{x} - \hat{\theta}\alpha = x^2 + y^2$

So
$$\tilde{x} = \frac{\hat{\theta}}{\hat{\theta}+\tilde{\theta}}(\tilde{x} + \tilde{\theta}\alpha) + \frac{\tilde{\theta}}{\hat{\theta}+\tilde{\theta}}(\tilde{x} - \hat{\theta}\alpha)$$
$$= \frac{\hat{\theta}}{\hat{\theta}+\tilde{\theta}}(x^1+y^1) + \frac{\tilde{\theta}}{\hat{\theta}+\tilde{\theta}}(x^2+y^2) = \underbrace{\left(\frac{\hat{\theta}}{\hat{\theta}+\tilde{\theta}}x^1 + \frac{\tilde{\theta}}{\hat{\theta}+\tilde{\theta}}x^2\right)}_{\text{Convex comb of BFSs}} + \underbrace{\left(\frac{\hat{\theta}}{\hat{\theta}+\tilde{\theta}}y^1 + \frac{\tilde{\theta}}{\hat{\theta}+\tilde{\theta}}y^2\right)}_{\text{Homogeneous soln}}$$

$\square$

<u>Theorem</u> Let K be the feasible region of (P). If (P) has a nondegenerate BFS, then $\dim(K) = n-m$.

<u>Proof</u> $\dim(K) \le n-m$: (P) has a nondegenerate BFS, so there are m lin. indep. columns of A. So $\{x : Ax=b\}$ has dimension $n-m$. So $\dim(K) \le n-m$

$\dim(K) \ge n-m$:

Generated Polyhedra vs. Constrained Polyhedra

Now $P = \{x : Ax \geq b\}$ is a polyhedron.

Also $Q = \{y : y = Dx$ for some $x\}$ is also a polyhedron.
(polys with other constraints on x)

P is defined in terms of <u>constraints</u>, Q is defined in terms of <u>generators</u>, i.e. everything in Q is a linear combination of the columns of D.

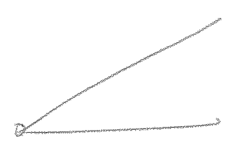
How are these two types of polyhedra related?

Will show (Weyl, Minkowski): they are the same

i.e., a constrained polyhedron can be written as a generated polyhedron, and vice versa.

Recall: K is a convex cone if $\left.\begin{array}{l} a^1, \dots, a^p \text{ in } K \\ \lambda_1, \dots, \lambda_p \geq 0 \end{array}\right\} \Rightarrow \lambda_1 a^1 + \dots + \lambda_p a^p$ in K.

(Note: K is a cone if $a \in K \Rightarrow \lambda a$ is in K for any $\lambda \geq 0$.

E.g.:  Just the lines)

Note: (1) Subspaces are convex cones
(2) If $\{K_i : i \in I\}$ is a family of convex cones then $\bigcap K_i$ is a convex cone.
(3) Any nonempty convex cone contains the origin
(4) Halfspaces of the form $\{x : a^T x \geq 0\}$ are cones.

Any subset $S \subseteq \mathbb{R}^n$ generates a convex cone:
$K(S) = \{\lambda_1 a^1 + \dots + \lambda_p a^p \quad p \geq 1, \lambda_i \geq 0, a^i \text{ in } S\}$
<u>Conical span</u> of S.

<u>Conical hull</u> of S: $\displaystyle\bigcap_{\substack{K \supseteq S \\ K \text{ convex cone}}} K$. Conical hull = conical span.

K is finitely constrained if $K = \{x : Ax \geq 0\}$ for some matrix A.
K is finitely generated if $K = \{x : x = Dy, y \geq 0\}$ for some matrix D.

For any $S \subseteq \mathbb{R}^n$, define the conical dual of S to be

$S^+ := \{x \text{ in } \mathbb{R}^n : a^T x \geq 0 \text{ for all } a \text{ in } S\}$. (Note: S^+ is often defined with $\leq$ sign. Here, use $\geq$ sign.)

When S is a cone, S^+ is the dual (polar) cone of S

Note: S^+ is always a convex cone – a constrained cone.

For S a finite set, S^+ is finitely constrained.

Prop Suppose S, T subsets of $\mathbb{R}^n$. Then

Do examples before proposition

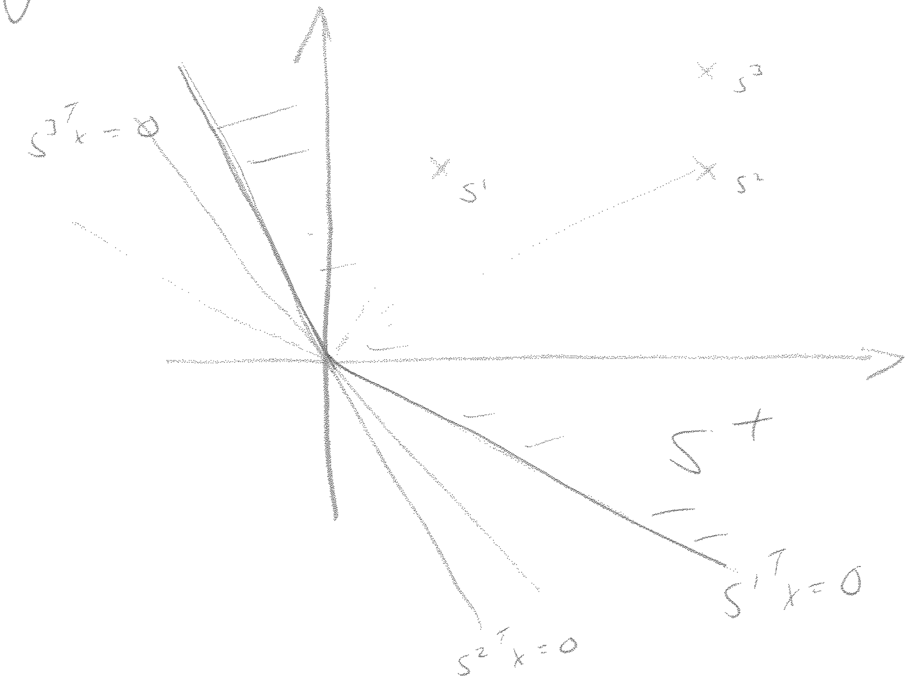
~~(1) $A \subseteq B \Rightarrow A$~~
(1) $S \subseteq T \Rightarrow T^+ \subseteq S^+$
(2) $S \subseteq S^{++}$ HW
(3) $S^+ = S^{+++}$ HW
(4) $S = S^{++} \Leftrightarrow S$ is a constrained cone HW
(5) If $T \subseteq S$ and T generates S conically then $T^+ = S^+$.

Proof (1) x in $T^+ \Rightarrow a^T x \geq 0$ for all a in $T \Rightarrow a^T x \geq 0$ for all a in $S \Rightarrow x$ in S^+
(2) $S^{++} = \{y : x^T y \geq 0 \text{ for all } x \text{ in } S^+\}$
~~If x is in S^+ then $x^T a \geq 0$~~
Let a be in S. If x is in S^+ then $a^T x \geq 0$.
ie $x^T a \geq 0$ for all x in S^+. Thus a is in S^{++}.
(3) $S^+ = S^{+++}$:
From (2), $S^+ \subseteq S^{+++}$. Need to show $S^{+++} \subseteq S^+$.
Let $z \in S^{+++}$. So $z^T y \geq 0$ for all y in S^{++}.
Since $S \subseteq S^{++}$, must have $z^T a \geq 0$ for all a in S.
Thus, $z \in S^+$.

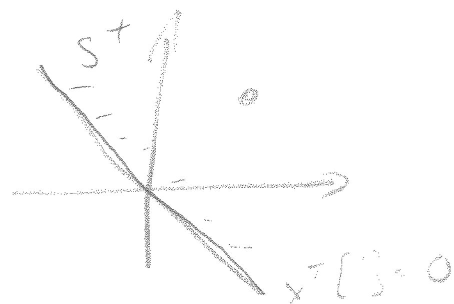
(4) $\Rightarrow$: $S = S^{++}$. Now $S^{++} = \{y : x^T y \geq 0 \text{ for all } x \text{ in } S^+\}$.
By definition, this is a constrained cone.
$\Leftarrow$: S is a constrained cone. $\therefore S = \{a : a^T b \geq 0 \text{ for all } b \text{ in some set } B\}$.
$\therefore S = B^+$. From (3), $B^+ = B^{+++}$. So $S = S^{++}$.

(5) ~~$S = \{x$~~ From (1), get $S^+ \subseteq T^+$. Need to show $T^+ \subseteq S^+$.
Let $a \in T^+$. So $a^T t \geq 0$ for all t in T.
Every point s in S can be expressed as a finite nonnegative combination of points in T: $s = \sum_{i=1} \lambda_i t^i$.
So $a^T s = \sum_{i=1} \lambda_i a^T t^i \geq 0$ since $\lambda_i \geq 0$, $a^T t^i \geq 0$. $\therefore a \in S^+$ $\square$

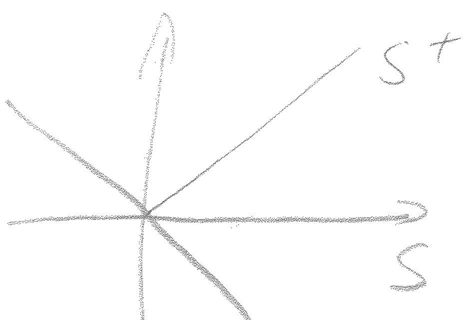
Eg: S consists of the three points $s^1 = \begin{bmatrix}1\\2\end{bmatrix}$, $s^2 = \begin{bmatrix}3\\2\end{bmatrix}$, $s^3 = \begin{bmatrix}3\\3\end{bmatrix}$.



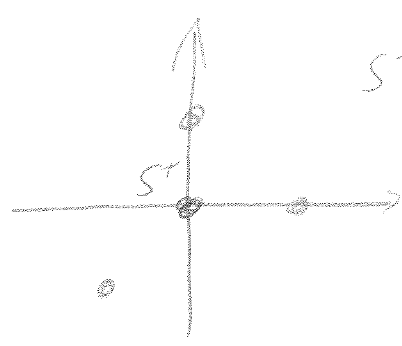
Eg: S consists just of the point $\begin{bmatrix}1\\1\end{bmatrix}$.



Eg: S consists of the ~~points~~ halfspace $x_1 + x_2 \geq 0$



Eg. S consists of the three points $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$, $\begin{bmatrix}-1\\-1\end{bmatrix}$.



S^+ is just the origin.

Fourier - Motzkin Elimination

System of inequality and equality relations

$$(\mathrm{I}) \qquad A^T y \begin{pmatrix} \geq \\ = \end{pmatrix} c$$

n variables $y_1, \ldots, y_n$ $\quad$ A is $m \times n$, c is in $\mathbb{R}^n$.

We eliminate y_n to create a new system

(II) linear equalities and inequalities now in variables $y_1, \ldots, y_{n-1}$.

Will have

(I) consistent $\Leftrightarrow$ (II) consistent.

Break into cases:

(0) If no coefficient of y_n is nonzero, take System(II) same as System(I).

Eg: $n=3$: (I) $\begin{array}{l} y_1 + y_2 = 2 \\ y_1 - y_2 \geq -2 \end{array}$ $\qquad$ (II) $\begin{array}{l} y_1 + y_2 = 2 \\ y_1 - y_2 \geq -2. \end{array}$

(1) There is some relation, say jth, which is an equality and in which y_n has a nonzero coefficient.

Then use that equation to get an expression for y_n ~~and~~ in terms of $y_1, \ldots, y_{n-1}$, and substitute into the other equations.

Eg: $n=3$ (I) $\begin{array}{rl} y_1 + y_2 - y_3 = -2 & (1) \\ 2y_1 + y_2 + y_3 \geq -1 & (2) \\ y_1 - y_2 \qquad = 1 & (3) \end{array}$ $\qquad$ #(1) $\Rightarrow y_3 = y_1 + y_2 + 2$

Then (2) becomes

$2y_1 + y_2 + (y_1 + y_2 + 2) \geq -1$

so (II) $\begin{array}{l} 3y_1 + 2y_2 \geq -3. \\ y_1 - y_2 = 1 \end{array}$

If not in case ~~(B)~~ (0) or (1), ~~all~~ all relations which have a nonzero coefficient in y_m are inequality relations.

(2a) Suppose all nonzero coeffs of y_m are positive, and y_m has a zero coeff in all equality relations.

Then just drop ~~y_m~~ all inequalities from (I) where y_m has a positive coeff.

Eg. (I)
$$\begin{aligned} y_1 + y_2 \quad\quad &= 2 \\ y_1 \quad\quad + y_3 &\geq 1 \\ -y_1 + 2y_2 + 3y_3 &\geq -2 \end{aligned}$$

(II) : $\quad y_1 + y_2 = 2.$

Why does this work?
If (I) is feasible, then clearly (II) is feasible – it is just a relaxation.
If (II) is feasible, then can make (I) feasible by taking y_m large enough.
Eg, in example, $y_1 = 0$, $y_2 = 2$ satisfies system (II).
If we then take y_3 so that $\quad 0 \quad + y_3 \geq 1$
and $-0 + 4 + 3y_3 \geq -2$
then system (I) is satisfied. So $y_3 \geq 1$ works.

b) Suppose all nonzero coeffs of y_m are negative, and y_m has a zero coeff in all equality relations.
Then just drop all inequalities from (I) where y_m has a negative coeff.

Eg: (I)
$$\begin{aligned} y_1 + y_2 \quad\quad &= 2 \\ y_1 \quad\quad - y_3 &\geq 1 \\ -y_1 + 2y_2 - 3y_3 &\geq -2 \end{aligned}$$

(II) $\quad y_1 + y_2 = 2.$

Works because: If $y_1, \ldots, y_{m-1}$ satisfies (II), can satisfy (I) by taking y_m to be negative enough.
Eg $y_1 = 0$, $y_2 = 2$: Need $-y_3 \geq 1$, $-3y_3 \geq -6$ ie $y_3 \leq -1$.

(3) All equalities have 0 coeff on y_m, and there exist a mixture of + and − coeffs on y_m in the inequalities.

Define (II) as follows:

(i) Retain all relations with 0 coefficient of y_m in (II).

(ii) For each ~~relation~~ pair of relations

$$a_{1i}y_1 + \cdots + a_{m-1,i}y_{m-1} + a_{mi}y_m \geq c_i, \quad a_{mi} > 0$$

$$a_{1j}y_1 + \cdots + a_{m-1,j}y_{m-1} + a_{mj}y_m \geq c_j, \quad a_{mj} < 0$$

these can be rewritten as:

$$i\text{th relation:} \quad y_m \geq \frac{1}{a_{mi}}\left(c_i - a_{1i}y_1 - \cdots - a_{m-1,i}y_{m-1}\right)$$

$$j\text{th relation:} \quad y_m \leq \frac{+1}{a_{mj}}\left(c_j - a_{1j}y_1 - \cdots - a_{m-1,j}y_{m-1}\right)$$

So certainly, in order to find y_m, need,

$$\frac{+1}{a_{mj}}\left(c_j - a_{1j}y_1 - \cdots - a_{m-1,j}y_{m-1}\right) \geq \frac{1}{a_{mi}}\left(c_i - a_{1i}y_1 - \cdots - a_{m-1,i}y_{m-1}\right).$$

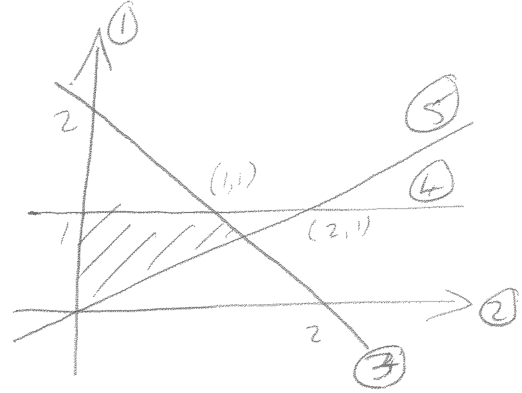
So include these relations in (II), and delete all relations where y_m has nonzero coeff.

So if there are p relations where $a_{mi} > 0$, q relations where $a_{mj} < 0$, replace these $p+q$ relations by pq new relations.

Example of Fourier-Motzkin elimination.

Eg: (I) $m=2$

$$\begin{array}{rcll} y_1 & \geq & 0 & (1) \\ y_2 & \geq & 0 & (2) \\ -y_1 - y_2 & \geq & -2 & (3) \\ -y_2 & \geq & -1 & (4) \\ -y_1 + 2y_2 & \geq & 0 & (5) \end{array}$$



(2) and (5) have $a_{i2} > 0$ (3) & (4) have $a_{i2} < 0$.

Get new relations from the pairs (2) & (3), (2) & (4), (5) & (3), (5) & (4).

(2) & (3): $\left.\begin{array}{l} y_2 \geq 0 \\ -y_1 - y_2 \geq -2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} y_2 \geq 0 \\ y_2 \leq 2 - y_1 \end{array}\right\}$ gives $0 \leq 2 - y_1$, ie $y_1 \leq 2$.

(2) & (4): $\left.\begin{array}{l} y_2 \geq 0 \\ -y_2 \geq -1 \end{array}\right\}$ gives $0 \leq 1$ ✓

(5) & (3) $\left.\begin{array}{l} -y_1 + 2y_2 \geq 0 \\ -y_1 - y_2 \geq -2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} y_2 \geq \frac{1}{2} y_1 \\ y_2 \leq 2 - y_1 \end{array}\right\}$ gives $\frac{1}{2} y_1 \leq 2 - y_1$
ie $y_1 \leq \frac{4}{3}$

(5) & (4): $\left.\begin{array}{l} -y_1 + 2y_2 \geq 0 \\ -y_2 \geq -1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} y_2 \geq \frac{1}{2} y_1 \\ y_2 \leq 1 \end{array}\right\}$ gives $\frac{1}{2} y_1 \leq 1$ ie $y_1 \leq 2$.

So (II) is:

$$\begin{array}{rcl} y_1 & \geq & 0 \\ -y_1 & \geq & -2 \\ -y_1 & \geq & -\frac{4}{3} \\ -y_1 & \geq & -2 \end{array}$$

If we now eliminated y_1, would get things

$\left.\begin{array}{l} 0 \leq 2 \\ 0 \leq \frac{4}{3} \\ 0 \leq 2 \end{array}\right\}$ all true.

So system is consistent.

Why does this work?

If $y_1, \dots, y_m$ satisfy (I), then the same $y_1, \dots, y_{m-1}$ will satisfy (II).

If $y_1, \dots, y_{m-1}$ satisfy (II):

Let $\alpha = \max_{i : a_{mi} > 0} \frac{1}{a_{mi}} (c_i - a_{1i} y_1 - \cdots - a_{m-1,i} y_{m-1})$

$\beta = \min_{j : a_{mj} < 0} \frac{+1}{a_{mj}} (c_j - a_{1j} y_1 - \cdots - a_{m-1,j} y_{m-1})$

~~Pick y_m to satisfy $\beta \le y_m \le \alpha$.~~

Since (II) is consistent, we know $\alpha \le \beta$.

So pick y_m to satisfy $\alpha \le y_m \le \beta$.

Then system (I) will be consistent.

Notice: if all rhs coefficients c_i are zero in system (I), then they are all zero in System (II).

EXAMPLE OF FOURIER-MOTZKIN ELIMINATION

Does the following system have a solution?

Use prescribed rules to eliminate variables

$$
\begin{array}{rcll}
y_1 + y_4 + y_5 & = & 2 & (1) \\
2y_1 + y_2 + y_3 - y_4 - y_5 & \geq & 1 & (2) \\
y_1 + y_3 & \geq & 5 & (3) \\
y_1 & \geq & 0 & (4) \\
y_2 & \geq & 0 & (5) \\
-y_1 - y_2 & \geq & -2 & (6) \\
-y_2 & \geq & -1 & (7) \\
y_1 + 2y_2 & \geq & 0 & (8)
\end{array}
$$

A) ~~If~~ Use eqn (1) to eliminate y_5:

(1) $\Rightarrow y_5 = 2 - y_1 - y_4$

Substitute into (2). Get:
System in y_1, y_2, y_3, y_4.

$$
\begin{array}{rcl}
3y_1 + y_2 + y_3 & \geq & 3 \\
y_1 + y_3 & \geq & 5 \\
\vdots & & \\
y_1 + 2y_2 & \geq & 0
\end{array}
$$

B) y_4 does not appear in any equation, so eliminate it.

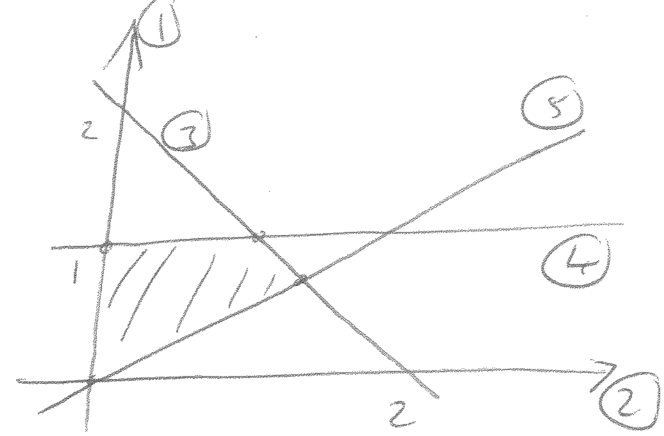
C) y_3 only appears in inequalities, and always with a positive sign.
So <u>drop</u> these inequalities.

(OK, because can choose y_3 large to satisfy these inequalities if we can choose y_1, y_2 to satisfy the others.)

(Can similarly drop inequalities if one variable always has a negative sign.)

D) Have:

$$\begin{aligned} y_1 &\geq 0 \quad (1) \\ y_2 &\geq 0 \quad (2) \\ -y_1 - y_2 &\geq -2 \quad (3) \\ -y_2 &\geq -1 \quad (4) \\ -y_1 + 2y_2 &\geq 0 \quad (5) \end{aligned}$$



To eliminate y_2:

Equations (2) & (3) give: $y_2 \geq 0, \quad y_2 \leq 2 - y_1$

So need $2 - y_1 \geq 0$

Any pair of inequalities where y_2 takes opposite signs gives an inequality.

(2) & (4) $\Rightarrow$ $y_2 \geq 0, \; y_2 \leq 1$. So need $0 \leq 1$.

(3) & (5) $\Rightarrow$ $y_2 \leq 2 - y_1, \; y_2 \geq \frac{1}{2}y_1$. So need $\frac{1}{2}y_1 \leq 2 - y_1$

(4) & (5) $\Rightarrow$ $y_2 \leq 1, \; y_2 \geq \frac{1}{2}y_1$. So need $\frac{1}{2}y_1 \leq 1$.

Thus, we have an equivalent system:

$$\left.\begin{aligned} y_1 &\geq 0 \\ 2 - y_1 &\geq 0 \\ 1 &\geq 0 \\ 2 - \tfrac{3}{2}y_1 &\geq 0 \\ 1 - \tfrac{1}{2}y_1 &\geq 0 \end{aligned}\right\} \text{Equivalently: } \left.\begin{aligned} y_1 &\geq 0 \\ -y_1 &\geq -2 \\ -\tfrac{3}{2}y_1 &\geq -2 \\ -\tfrac{1}{2}y_1 &\geq -1 \end{aligned}\right\}$$

Eliminating y_1 in the same way gives

$0 \leq 2$

$0 \leq \frac{4}{3}$

$0 \leq 2$

If y_1 satisfies all these, then we can find y_2 satisfying the system at the top of the page.

All true. So system is <u>consistent</u>.

Weyl's Theorem

Any nonempty, finitely generated cone is polyhedral.

Proof $K = \{ Ax : x \geq 0 \}$ for some A in $\mathbb{R}^{m \times n}$.

$$= \left\{ y : \begin{pmatrix} y - Ax = 0 \\ x \geq 0 \end{pmatrix} \text{ is a consistent system in } (x, y) \right\}$$

Now, use Fourier-Motzkin elimination to eliminate x. Get

$$K = \{ y : Dy \geq 0 \} \text{ for some matrix } D. \quad \square$$

~~Corollary~~

~~Corollary 1 Among all subsets S of $\mathbb{R}^n$ st~~

Corollary 1 Given A in $\mathbb{R}^{m \times n}$, let $K = \{ Ax : x \geq 0 \}$

$L = \{ y : A^T y \geq 0 \}$. $L = \{ y : A^T y \leq 0 \}$

Then (i) $K^+ = L$ (ii) $L^+ = K$. (polar L corresponds to negative dual of polar cone,

Proof (i) $K^+ = \{ z : z^T w \overset{\leq}{\geq} 0 \text{ for all } w \text{ in } K \}$

$= \{ z : z^T A x \overset{\leq}{\geq} 0 \text{ for all } x \geq 0 \}$

$= \{ z : (A^T z)^T x \overset{\leq}{\geq} 0 \text{ for all } x \geq 0 \}$.

So $y \in L \Rightarrow A^T y \overset{\leq}{\geq} 0 \Rightarrow y \in K^+$, so $L \subseteq K^+$.

Also, $z \in K^+ \Rightarrow A^T z \overset{\leq}{\geq} 0$ since if $A^T z$ has a ~~negative~~ positive component,

choose $x = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

$\Rightarrow z \in L$, so $K^+ \subseteq L$.

Thus, $K^+ = L$.

(ii) By Weyl's Theorem, $K = \{ w : D^T w \geq 0 \}$ for some D. So, by earlier proposition, $K = K^{++}$. Now $L^+ = (K^+)^+$ from (i), so $L^+ = K$ $\square$

Corollary #2 **Farkas's Lemma**

Exactly one of the following systems has a solution:

(I) $Ax = b$, $x \geq 0$ (II) $A^T y \overset{\leq}{\geq} 0$, $b^T y \overset{\geq}{<} 0$

Proof (I) fails $\Rightarrow b \notin K := \{Ax : x \geq 0\} \xRightarrow{\text{by Weyl's Theorem}} b \notin K^{++}$

~~$\Rightarrow \exists y$ in K^+ with $b^T y < 0$~~

$\Rightarrow \exists y$ in $K^+ = \{y : A^T y \overset{\leq}{\geq} 0\}$ from Corollary 1 with $b^T y \underset{>}{\leq} 0$

$\Rightarrow$ (II) holds.

(II) holds $\Rightarrow \exists y$ in $L = \{y : A^T y \overset{\leq}{\geq} 0\}$ with $b^T y \overset{\geq}{<} 0$

$\Rightarrow b \notin L^+ \Rightarrow b \notin K = \{Ax : x \geq 0\}$ from Corollary 1

$\Rightarrow$ (I) fails $\square$

Corollary 3 **Minkowski's Theorem**

Any finitely constrained cone is nonempty and finitely generated.

Proof Let $L = \{y : A^T y \overset{\leq}{\geq} 0\}$. Clearly, the origin is in K, so $K \neq \emptyset$.

By Corollary 1, $L^+ = K = \{Ax : x \geq 0\}$

By Weyl's Theorem, $K^+ = \{z : D^T z \overset{\leq}{\geq} 0\}$ for some D.

By Corollary 1, $L = K^+ = \{Du : u \geq 0\}$ $\square$

Extend the conical results to polyhedra.

(Affine Weyl) Theorem (finitely generated polyhedra are finitely constrained.)

Let $P = \{x : x = By + Cz,\ y \geq 0,\ z \geq 0,\ \sum_{i=1}^{q} z_i = 1\} \subseteq \mathbb{R}^n$
with B in $\mathbb{R}^{n \times p}$, C in $\mathbb{R}^{n \times q}$.

Then $\exists$ matrix A in $\mathbb{R}^{m \times n}$ and b in $\mathbb{R}^m$ with

$P = \{x \text{ in } \mathbb{R}^n : Ax \geq b\}$.

Proof (Technique called HOMOGENIZATION.)

If $P = \phi$: (So B and C are vectors) Take $A = [0 \ldots 0]$, $b = 1$.

Otherwise, $P \neq \phi$:

Define $P' \subseteq \mathbb{R}^{n+1}$ by

$$P' = \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & 1 \ldots 1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix},\ y \geq 0,\ z \geq 0 \right\}.$$

This is a cone, and $P = \left\{ x : \begin{bmatrix} x \\ 1 \end{bmatrix} \text{ is in } P' \right\}$,

ie, P is a slice through the cone.

By Weyl's Theorem,

$$P' = \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : [A \mid d] \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} \geq 0 \right\} \text{ for some } [A \mid d] \text{ in } \mathbb{R}^{m \times (n+1)}.$$

Let $b = -d$. ~~Then A~~ Then

~~$P' = \{$~~

$$P = \left\{ x : \begin{bmatrix} x \\ 1 \end{bmatrix} \text{ is in } P' \right\} = \left\{ x : [A \mid -b] \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0 \right\} = \{x : Ax - b \geq 0\}$$

$$= \{x : Ax \geq b\}$$ $\square$

<u>(Affine Minkowski) Theorem</u> (Finitely constrained polyhedra are finitely generated.)

Suppose $P = \{x \text{ in } \mathbb{R}^n : Ax \geq b\}$ where A in $\mathbb{R}^{m \times n}$, b in $\mathbb{R}^m$. Then $\exists$ there exist matrices B in $\mathbb{R}^{n \times p}$, C in $\mathbb{R}^{n \times q}$ such that

$$P = \{x : x = By + Cz,\ y \geq 0,\ z \geq 0,\ \sum_{i=1}^{q} z_i = 1\}.$$

<u>Proof</u> If $P = \phi$, take $p = q = 0$, ie B, C vacuous.

Otherwise, $P \neq \phi$:

Again, homogenize. Consider

$$P' = \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : \begin{bmatrix} A & -b \\ 0 \cdots 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} \geq 0 \right\} \qquad (*)$$

Again, x is in $P \iff \begin{bmatrix} x \\ 1 \end{bmatrix}$ is in P'.

By Minkowski's Theorem,

$$\cancel{P' = \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : D \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} \geq 0 \right\} \text{ for some } D \text{ in } \mathbb{R}^{m \times (n+1)}}$$

$$P' = \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} = D\omega,\ \omega \geq 0 \right\} \text{ for some } D \text{ in } \mathbb{R}^{(n+1) \times m}.$$

We can rearrange the columns of D if necessary so that

$$D = \begin{bmatrix} B & C \\ 0 \cdots 0 & \neq 0 \cdots \neq 0 \end{bmatrix}. \qquad \text{Take } \omega = \begin{bmatrix} y \\ z \end{bmatrix}. \quad \begin{matrix} B \in \mathbb{R}^{n \times p} & y \in \mathbb{R}^p \\ C \in \mathbb{R}^{n \times q} & y \in \mathbb{R}^q. \end{matrix}$$

So $P' = \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} B & C \\ 0 \cdots 0 & \neq 0 \cdots \neq 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix},\ y \geq 0, z \geq 0 \right\}.$

Want to pin down the nonzeroes in the bottom row of D.

Now, if any of the nonzeroes are negative, can choose z so that element of $z = 1$, all other element of z are zero. This will give $x_{n+1} < 0$.

But we know that in P', from (*), must have $x_{n+1} \geq 0$.

Thus all the nonzeroes in the last row of D are > 0. We can rescale the columns of C if necessary so that all the nonzeroes are 1. This gives:

$$P' = \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} B & C \\ 0 \cdots 0 & 1 \cdots 1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}, y \geq 0, z \geq 0 \right\}.$$

Now $P = \left\{ x : \begin{bmatrix} x \\ 1 \end{bmatrix} \text{ is in } P' \right\}$

$$= \left\{ x : x = By + Cz,\ y \geq 0,\ z \geq 0,\ \sum_{i=1}^{a} z_i = 1 \right\}.$$

Together, the Affine Weyl and Affine Minkowski theorems are called the Double Decomposition Theorem.

Can be extended to:

<u>Goldman's Resolution Theorem</u> Suppose $P = \{x : Ax \geq b\} \neq \phi$, polyhedron.

Then $P = S + K + Q$ where

$S = \{x \text{ in } \mathbb{R}^n : Ax = 0\}$ ~~(= lineality space)~~

$S + K = \{x \text{ in } \mathbb{R}^n : Ax \geq 0\}$

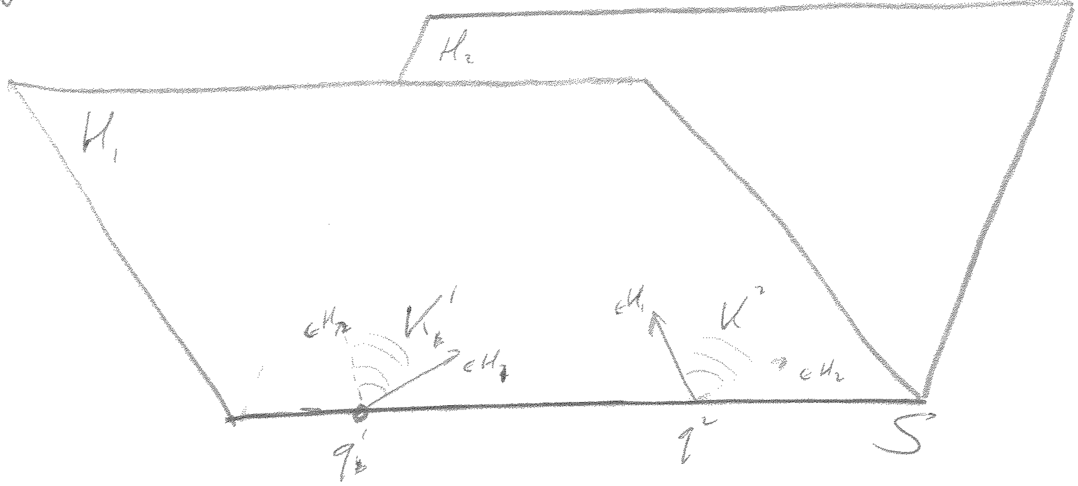
K is a pointed cone

$K + Q$ is a pointed polyhedron.

Q is a polytope given by the convex hull of extreme points of $K + Q$.

(ie $x \in P \iff x = s + k + q$ where $s \in S, k \in K, q \in Q$.)

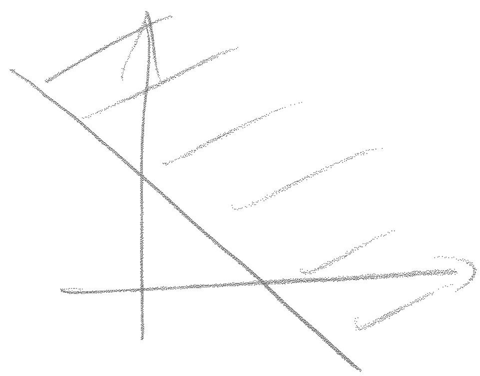
Eg: S and $S+K$ are unique, but K and Q need not be unique.



P = Span between H_1 and H_2.

A polyhedron is pointed if it contains an extreme point.

$P = \{x \in \mathbb{R}^2 : x_1 + x_2 \geq 3\}$ is not pointed.



$$P = \{x : x_1 + x_2 = 0\} + \left\{\lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix} : \lambda \geq 0\right\} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$$= \{x : x_1 + x_2 = 0\} + \left\{\lambda \begin{bmatrix} 1 \\ 3 \end{bmatrix} ; \lambda \geq 0\right\} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

Farkas $\Rightarrow$ Strong Duality:

$\min\{c^T x : Ax \geq b\}$

$= \min\{c^T x : x = By + Cz,\ y \geq 0,\ z \geq 0,\ \sum z_i = 1\}$.

If min exists, achieved by one of cols of C.

" of dual max exists it is achieved

~~Farkas~~

~~Let~~

$$\begin{array}{llll} \min & c^T x & & \max\ b^T y \\ & Ax = b & (P) & A^T y \leq c \quad (D) \\ & x \geq 0 & & \end{array}$$

Optimal x^* $\quad y^*$

? $c^T x^* = b^T y^*$? Let $b^T y^* = \pi^*$

Consider

$$A' = \begin{bmatrix} A & 0 \\ c^T & 1 \end{bmatrix} \qquad b' = \begin{bmatrix} b \\ \pi^* \end{bmatrix}$$

Farkas: (I) $\begin{bmatrix} A & 0 \\ +c^T & 1 \end{bmatrix}\begin{bmatrix} x \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} b \\ \pi^* \end{bmatrix}$ $\quad$ (II) $\begin{bmatrix} A^T & +c \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y \\ y_{m+1} \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_{n+1} \geq 0$ $\quad$ $b^T y + \pi^* y_{m+1} > 0$

Look at (II) $y_{m+1} = 0 \Rightarrow A^T y \leq 0$ if $b^T y > 0$ contradicting π^*

So need $y_{m+1} < 0$. So $A^T y \leq -c\, y_{m+1}$ Rescale: $A^T y' \leq c$, $b^T y' - \pi^* > 0$ Impossible
$y' = y/{-y_{m+1}}$ (II) when π^* max.