Infinitesimal generator $A$ plays a fundamental role in the Markov process framework (including for SDE's).

$$A^{(t)} = v(x,t) \frac{\partial}{\partial x} + K(x,t) \frac{\partial^2}{\partial x^2}$$

This is an operator which acts on functions of $x$, and it is parameterized by time $t$. Important special case is the time-homogenous case when the SDE is autonomous and so the drift and diffusion coefficients are time-independent and so the infinitesimal generator also does not depend on time. This infinitesimal generator is like a continuous-space analogue to the transition rate matrix for a continuous-time Markov chain. You can see this by taking a finite difference approximation to the infinitesimal generator. See the Majda and Kramer notes on "Random Method of Characteristics".

The backward and forward Kolmogorov equation can be expressed simply in terms of $A$:

**Backward**: $-\frac{\partial \rho}{\partial t} = A^{(s)}_x \rho$

**Forward**: $\frac{\partial \rho}{\partial t} = (A^{(e)}_y) \rho$

How can one use the backward and forward Kolmogorov equations to answer useful questions about SDEs?
What is the probability distribution (density) for the state of the system at a future time $t$, given some initial probability distribution.

$$\text{Prob}(X(t) \in B) = \int_B \varnothing(x, t) \, dx$$

For $X(t)$

$$\text{Prob}(X(t) \in B) = \int_{\mathbb{R}} \text{Prob}(X(t) \in B \mid X(0) = x) \, \varnothing(x, 0) \, dx$$

Law of total probability

Prescribe $\varnothing(x) = \varnothing(x, 0)$

$$\varnothing(x, t) = \int_{\mathbb{R}} \varrho(0, x, t, \gamma) \, \varnothing(\gamma) \, d\gamma$$

The probability transition density appears like a Green's function (fundamental solution) for the probability density of the state of the system at a given time.

$$\frac{\partial \varnothing(x, t)}{\partial t} = (A_x^t) \varnothing(x, t)$$

$\varnothing(x, 0) = \varnothing(x)$

The probability density for the state of a system obeys the forward Kolmogorov or Fokker-Planck equation.

A particular question of wide importance is: Does the state of the system evolve toward a limiting probability distribution.

$$\varnothing(\gamma) = \lim_{t \to \infty} \varnothing(\gamma, t)$$
distribution.

\[ \overline{\mathbf{\theta}}(y) = \lim_{t \to a} \overline{\mathbf{\theta}}(y, t) \]

Proving this happens is hard (check ergodicity...), but on a practical level one often tries to argue that a limit distribution should exist on plausible grounds. If a limit distribution exists, it has to be a stationary distribution, and therefore a time-independent solution to the forward Kolmogorov equation. This limit distribution only really makes sense in the time-homogenous case.

\[ A_y \overline{\theta} = 0 \]

boundary conditions....

Example of microparticle:

\[ d\mathbf{x} = \mathbf{v} \, dt \]

\[ \mathbf{v} = -\frac{\partial}{\partial x} \mathbf{v} \, dt + \sqrt{\frac{2k_B T}{m}} \, dW(t) \]

**Infinitesimal generator:**

\[ D_{inf} \mathbf{v} = \begin{bmatrix} \mathbf{v} \\ -\frac{\partial v}{\partial y} \end{bmatrix} \]

**Diffusion**

\[ K = \begin{bmatrix} 0 & 0 \\ 0 & \frac{k_B T}{m^2} \end{bmatrix} \]

\[ A = \begin{bmatrix} \mathbf{v} \\ -\frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{2}{\partial x} \end{bmatrix} \]

\[ + \begin{bmatrix} 0 & 0 \\ 0 & \frac{k_B T}{m^2} \end{bmatrix} \begin{bmatrix} \frac{2}{\partial x} \\ \frac{2}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{2}{\partial x} \\ \frac{2}{\partial y} \end{bmatrix} \]
What is the stationary probability distribution for this system? The X dynamics don't settle down to a stationary probability distribution (not positive recurrent) so let's ignore X for now and just ask, what is the stationary probability distribution for V?

The truncated infinitesimal generator that describes only the V dynamics is:

\[
A = -\frac{kv}{m} \frac{\partial}{\partial v} + \frac{k_b T v}{m^2} \frac{\partial^2}{\partial v^2}
\]

Stationary distribution satisfies:

\[
A \phi = 0
\]

\[
\frac{2}{m} \left( \frac{v \phi}{m} \right) + \frac{1}{m^2} \frac{\partial^2}{\partial v^2} \left( \frac{k_b T v \phi}{m^2} \right) = 0
\]

Involves one fewer dimension so sometimes solvable when the time-dependent forward Kolmogorov is not analytically solvable. In this example, the equation is one-dimensional so can be solved by quadrature.

\[
\frac{\partial \phi}{\partial v} = -\frac{kv}{k_b T} \phi + c_i = 0 \quad \text{for} \quad 0 \leq v \leq \infty
\]

Integrating for \(v\):

\[
\phi(v) = c_1 \exp \left( -\frac{mv}{k_b T} \right)
\]

\[
= c_2 e^{-\frac{1}{2} \frac{mv^2}{k_b T}}
\]

\[
\phi(\infty) = c_2 e^{-\frac{1}{2} \frac{mv^2}{k_b T}}
\]
If we had included a potential energy term, then solving the SDE’s would not be possible analytically (if the potential energy were not quadratic), but one could still solve the stationary distribution from the time-independent Fokker-Planck equation to show that the Gibbs distribution is a stationary distribution.

Applications of the backward Kolmogorov equation:

Suppose we want to calculate:

\[ u(s, x) = \mathbb{E} \left[ f(\mathbb{X}(t)) \mid \mathbb{X}(s) = x \right] \]

Given: \( f \) is like a payoff or cost function

set \( t \) is fixed

What is the expected cost or payoff in the future given an earlier state of system.
\[ u(s, x) = \int_{\mathbb{R}} f(y) \, p(s, x; t, y) \, dy \]

\[ -\frac{\partial u(s, x)}{\partial s} = A_x^{(s)} \, u(s, x) \]

\[ u(s = t, x) = f(x) \quad \text{"final condition"} \]

Note that the Kolmogorov backward equation is sometimes formulated with respect to forward time when the SDE is time-homogenous.

For time-homogenous situation the answer to the above question is only a function of \( t' = t - s \)

\[
\begin{align*}
S_0 \quad & -\frac{2}{\partial s} \rightarrow \frac{2}{\partial t} \\
& -\frac{2}{\partial s} = \frac{2}{\partial t} = \frac{2}{\partial t'}
\end{align*}
\]

\[
S_0 \quad u(t - s, x) = \mathbb{E} \left( f(X(t)) \bigg| X(s) = x \right)
\]

\[
\frac{\partial u(t, x)}{\partial t} = A \, u(t, x)
\]

Quick word on PDEs

(\text{Kramer + Majda, } \text{"Diffus Run", \"Method of Characteristics\"

\"Random Method of Characteristics\")

Consider constant-coef. backward Kolmogorov

\[
\frac{\partial u}{\partial t} + \sqrt{\frac{\partial u}{\partial x}} + K \frac{\partial^2 u}{\partial x^2}
\]
\[
\frac{\partial u}{\partial t} = \nu \frac{\partial^2 x}{\partial t^2}
\]

\[u(x, t=0) = f(x)\]

\[u(x, t) = \int_{-\infty}^{\infty} \exp \left( -\frac{(x + v \sqrt{t} - x)^2}{2 \sqrt{4\pi kt}} \right) f(x) dx\]

Greens function

Variable-coef case:
Same idea but harder to get
analytic solution

W/o diffusion:
\[
\frac{\partial x(u, t)}{\partial t} = \nu(x(t)) \frac{\partial u(x, t)}{\partial x}
\]

\[u(x, t=0) = f(x)\]

First order PDE; solve by Method of Characteristics
Express solution of PDE in terms of solution to
\[
ODE \quad \frac{dx}{dt} = -\nu(x(t))
\]

Backward Kolmogorov equation associated
Second order PDE
\[ \frac{\partial u(x,t)}{\partial t} = v(x,t) \frac{\partial u(x,t)}{\partial x} + K(x,t) \frac{\partial^2 u}{\partial x^2} \]

\[ u(x, t = 0) = f(x) \]

This is equivalent to a

SDE: (random character)\n
\[ dX = v(x,t) \, dt + \sqrt{2K(x,t)} \, dW \]