Homework 3 is available online, due Friday, April 20.

Homework 4 will be due Wednesday, May 2.

We now approach SDE's from the framework of Markov processes.

A stochastic process \( \mathcal{X}(t) \) is a Markov process with respect to a filtration \( \mathcal{F}_t \) provided:

\[
\Pr(\mathcal{X}(t) \in B | \mathcal{F}_s) = \Pr(\mathcal{X}(t) \in B | \mathcal{X}(s))
\]

for \( s \leq t \)

There are fundamental descriptors of a Markov process:

- Transition probability function

\[
P(s, x; t, B) = \Pr(\mathcal{X}(t) \in B | \mathcal{X}(s) = x)
\]

If the dynamics of the Markov process are not explicitly time dependent (they behave like an autonomous differential equation), then the Markov process is said to be time-homogenous and the transitional probability function depends only on the difference between the two times \( t - s \).

The transition probability function can be expressed as a measure with respect to its target spatial argument:

\[
P(s, x; t, dy) = \int_B P(s, x; t, dy)
\]

Often in practice, one can define a transition probability density.
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\[ p(s, x; t, y) \]  
(or \[ p(t, y | s, x) \])

\[ P(s, x; t, dy) = p(s, x; t, y) dy \]

(Careful: Some time need to allow for \( \delta \)-fn contributions)

Objective: Solutions of SDE's are Markov processes. (Crucial: The noise is independent over different time intervals. In some more general stochastic evolution equations (fractional Brownian motion) may not be Markov processes and one will get instead more complicated integro-differential equations for the probability transition density.)

We now ask the question: how can we compute the probability transition (density)? We will show that the answer can be given in terms of solutions of certain deterministic partial differential equations.

We will answer this question by using the framework of Diffusion Processes as a link between SDE's and Markov processes.

Diffusion process is a Markov process with the following properties: For any fixed \( \varepsilon > 0 \):

1) \[ \lim_{t \to 0} \frac{1}{t - s} \sum_{|y - x| > \varepsilon} P(s, x; t, dy) = 0 \]

\[ \Pr (|X(t) - X(s)| > \varepsilon | X(s) = x) \]  
(exclude jumps)

2) \[ \lim_{t \to 0} \frac{1}{t - s} \sum_{|y - x| < \varepsilon} (y - x) P(s, x; t, dy) \]

\[ = \nabla(X, s) \]  
(drift)

Intuitively, can forget about \( \varepsilon \) in the limit ...)
To check condition (i), it is sufficient to check:

\[ \lim_{t \to s} \frac{1}{t-s} \left\{ \mathbb{E} \left( \frac{|X(t) - x|^2}{2(t-s)} \bigg| X(s) = x \right) \right\} \]

Intuitively, one can forget about \( \varepsilon \) and check:

\[ \lim_{t \to s} \frac{1}{2(t-s)} \int_{|y-x| < \varepsilon} (y-x)^2 p(x, t, dy) \]

\[ = K(x, s) \quad ( \text{diffusivity} ) \]

One can generalize diffusion processes directly to multiple dimensions: drift becomes a vector function and the diffusivity becomes a tensor (matrix function).

Technical comment: To check condition (i), it is sufficient to check:

\[ \lim_{t \to s} \int_{-\infty}^{\infty} \frac{1}{t-s} \left( |x-y|^2 + s \right) p(x, s; t, dy) = 0 \]

for some \( s > 0 \).

We now sketch how one can show how stochastic...
differential equations are connected to diffusion processes. Once we make that linkage, we will develop general Markov process theory for diffusion processes, and this will give us concrete equations for the probability transition density associated to SDE's.

How are SDE's related to diffusion processes, which are a special important case of Markov processes?

\[
dX = a(X,t) \, dt + b(X,t) \, dW(t), \quad X(t=0) = X_0
\]

One can show that the solutions to such an SDE (under technical conditions on the coefficients) can be described by a diffusion process with

**drift:** \( \nu(X,s) = a(X,s) \)

**diffusivity:** \( K(X,s) = \frac{1}{2} b^2(X,s) \)

**Sketch of how how:**

1. Use techniques similar to those to derive estimates for the solution of stochastic differential equation as were used in the proof of existence of solutions to SDE's

\[
\left| \mathbb{E} \left( \left( X(t) - X(s) \right)^4 \mid X(s) = x \right) \right| \leq C \left| t - s \right|^2
\]

and use the technical comment.

\[ \hat{X}(t) = \hat{X}(s) + \int_{s}^{t} a(\hat{X} \omega) \, d\omega + \int_{s}^{t} b(\hat{X} \omega) \, dW(\omega) \]

\[
\mathbb{E}(X(t) - x \mid X(s) = x) = \int_{s}^{t} \mathbb{E}(a(\hat{X}(\omega)) \mid X(s) = x) \, d\omega
\]

\[ u = s + \theta(t-s) \]

\[
= (t-s) \int_{0}^{\theta(t-s)} \mathbb{E}
\]
\[ \lim_{t \to s} \mathbb{E} \left( \frac{X(t) - x}{t-s} \right) | X(s) = x \]

\[ = \lim_{t \to s} \mathbb{E} \left( \mathbb{E} \left( \frac{X(t) + \theta(t-s)}{t-s} \right) | X(s) = x \right) \]

\[ = \lim_{t \to s} \mathbb{E} \left( \mathbb{E} \left( \frac{X(t)}{t-s} \right) | X(s) = x \right) \]

And then one has to fuss with the \( \Sigma \).

\[ \mathbb{E} \left( \frac{(X(t) - x)^2}{(t-s)^2} \right) | X(s) = x \]

\[ = \mathbb{E} \left( \left( \int_s^t \mathbb{E} \left( \mathbb{E} \left( X(u) | X(s) = x \right) \right) du + \int_s^t b(X(u) \mathbb{E} \left( X(u) | X(s) = x \right) du \right)^2 \right) \right) \]

\[ = \mathbb{E} \left( \left( \int_s^t \mathbb{E} \left( X(u) | X(s) = x \right) du \right)^2 \right) \]

\[ = \mathbb{E} \left( \left( \int_s^t b(X(u) \mathbb{E} \left( X(u) | X(s) = x \right) du \right)^2 \right) \]

\[ = \lim_{t \to s} \mathbb{E} \left( \frac{(X(t) - x)^2}{2(t-s)} \right) | X(s) = x \]

\[ = \lim_{t \to s} \mathbb{E} \left( \frac{\left( \int_s^t b(X(u) \mathbb{E} \left( X(u) | X(s) = x \right) du \right)^2}{2(t-s)} \right) | X(s) = x \]
Diffusion process framework gives an alternative, sometimes helpful perspective on the stochastic dynamics. It suggests how one can derive drift and diffusion coefficients from data on an observed time series of a state function which one wants to model with an SDE.

Now we have embedded SDE's into diffusion process theory, which is itself a special case of Markov process theory.

Now we will develop some general Markov process theory and see how it applies to SDE's.

**Chapman-Kolmogorov equation**

For $s < u < t$

$$P(s, x; t, dy) = \int P(s, x; u, dz) P(z, y; t, dy)$$

**Proof:**

$$P(s, x; t, B) = \Pr(\mathbb{X}(t) \in B | \mathbb{X}(s) = x)$$

$$= \int_{-\infty}^{\infty} \Pr(\mathbb{X}(t) \in B, \mathbb{X}(u) \in dz | \mathbb{X}(s) = x)$$

(law of total prob)
\[
\begin{align*}
= \int_{-\infty}^{\infty} \Pr \left( X(t) \in B \mid X(u) \in d_2, X(s) = x \right) \\
\quad \cdot \Pr \left( X(u) \in d_2 \mid X(s) = x \right) \\
\quad \cdot \Pr \left( A \cap B \mid C \right) = \Pr(A \mid B \cap C) \Pr(B \mid C)
\end{align*}
\]

\[
\begin{align*}
= \int_{-\infty}^{\infty} \Pr \left( X(t) \in B \mid X(u) \in d_2 \right) \\
\quad \cdot \Pr \left( X(u) \in d_2 \mid X(s) = x \right) \\
\quad \cdot \Pr \left( A \cap B \mid C \right) = \Pr(A \mid B \cap C) \Pr(B \mid C)
\end{align*}
\]

\[
\begin{align*}
= \int_{-\infty}^{\infty} \Pr(u, z, t, B) \\
\quad \cdot \Pr(s, x, u, d_2)
\end{align*}
\]