Homework 3 posted soon (tomorrow?).

No office hours Wednesday March 28.

\[ f \text{ is Hölder continuous with } \alpha \text{ if } \exists \beta > 0 \quad \text{s.t.} \quad |t-s| < \beta \quad \Rightarrow \quad |f(t) - f(s)| \leq C \cdot |t-s|^{\alpha} \]

Holder condition

\[ \lim_{t-s \to 0} \frac{f(t) - f(s)}{t-s} = 0 \]

Lipschitz theorem for functions with Hölder exponents \( \alpha > 1 \),

\[ \frac{f(t) - f(s)}{t-s} \leq C \cdot |t-s|^{\alpha-1} \to 0 \]

Numerical analysis of stochastic differential equations:

Kloeden and Platen, Numerical Solution of Stochastic Differential Equations: rigorously

We'll study orders of accuracy on a more formal intuitive level.

The starting point is Itô's formula, which is used to develop stochastic Taylor expansions (Kloeden and Platen, Ch. 5)

\[ d\mathbf{x} = a(\mathbf{x}(t), t) \, dt + b(\mathbf{x}(t), t) \, dW(t) \]

\[ df(\mathbf{x}(t), t) = \left( \frac{\partial f}{\partial t} + a(\mathbf{x}(t), t) \frac{\partial f}{\partial x} + \frac{1}{2} b^2(\mathbf{x}(t), t) \frac{\partial^2 f}{\partial x^2} \right) \, dt + b(\mathbf{x}(t), t) \, dW \]
We will use this integrated form of Ito's formula to develop a stochastic Taylor expansion valid through first order by simply evaluating the integrands at the beginning of the time interval $t$. To get higher order stochastic Taylor expansion, use Ito's formula on the integrands to get iterated (stochastic) integrals.

For our purposes, it will suffice to write:

\[
\begin{align*}
&f(X(t'), t) - f(X(t), t) \\
&= \int_t^{t'} \left( \frac{\partial f}{\partial t} (X(s), t) + \frac{\partial f}{\partial x} (X(s), t) \frac{\partial X}{\partial x} (X(s), t) \\
&\quad + \frac{1}{2} b^2 (X(s), t) \frac{\partial^2 f}{\partial x^2} (X(s), t) \right) ds \\
&\quad + \int_t^{t'} b(X(s), t) \frac{\partial f}{\partial x} (X(s), t) \frac{\partial X}{\partial w} (X(s), t) \frac{\partial w}{\partial s} (s) ds \\
&\quad + \int_t^{t'} \left[ g(X(s), t) - g(X(t'), t') \right] ds \\
&\quad + \int_t^{t'} \left[ h(X(s), t) - h(X(t'), t') \right] dW(s)
\end{align*}
\]
\[ g = \frac{\partial f}{\partial t} + a \frac{\partial f}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2 f}{\partial x^2} \]

\[ h = b \frac{\partial f}{\partial x} \]

Take \( t = t' \leq t + \alpha t \); \( \alpha t \) small,

\[ f(\mathbf{x}(t'), t') - f(\mathbf{x}(t), t) = b \left[ \mathbf{x}(t', t) \right] \frac{\partial f}{\partial x} \left( \mathbf{x}(t'), t \right) \left( \mathbf{x}(t') - \mathbf{x}(t) \right) + O(\alpha t) \]

We will use this result for \( f = 0 \), \( f \geq b \) in expression for \( \delta \mathbf{x} \):

\[ \mathbf{x}(t + \alpha t) - \mathbf{x}(t) = \int_{t}^{t + \alpha t} a(\mathbf{x}(s), t) \, ds \]

\[ + \int_{t}^{t + \alpha t} b(\mathbf{x}(s), t) \, dW(s) \]

Euler-Maruyama:

\[ \mathbf{x}_{\text{em}}(t + \alpha t) - \mathbf{x}(t) = \int_{t}^{t + \alpha t} a(\mathbf{x}(t), t) \, ds \]

\[ + \int_{t}^{t + \alpha t} b(\mathbf{x}(t), t) \, dW(s) \]

\( \mathbf{x}(t + \alpha t) - \mathbf{x}(t) \approx (t + \alpha t) \)
Martingales can be thought of intuitively as behaving like sums of independent mean zero, random variables. In particular, there are rigorous central limit theorems for martingales. What this means for us:

Suppose we are solving an SDE with a numerical method
over a time interval of size $T$, using time step $\Delta t$ and $N = \frac{T}{\Delta t}$ time steps. How do the errors from each time step accumulate when we get to time $T$.

Normal errors without special properties: just multiply the error in each step by $N$.

Martingale errors: multiply by $\sqrt{N}$ by a sort of martingale central limit theorem which generalizes the notion that a sum of $N$ independent mean zero random variables will have variance scaling like $N$ so standard deviation (size) scales as $\sqrt{N}$.

So global error for Euler-Marayama method:

$$\sqrt{N}O(\Delta t) + \sqrt{N}O(\Delta t^{3/2}) + N(\Delta t)^2$$

$$\approx \sqrt{\frac{T}{\Delta t}} O(\Delta t^{1/2}) + \sqrt{\frac{T}{\Delta t}} O(\Delta t^{3/2}) + \frac{T}{\Delta t} O((\Delta t)^2)$$

$$= O(\sqrt{T/\Delta t}) + O(T/\Delta t)$$

EM is half order accurate in strong sense

Weak accuracy:

$$|E g(X(t + \Delta t)) - E g(X_{em}(t + \Delta t))|$$

$$= |E (g(X(t + \Delta t)) - g(X_{em}(t + \Delta t)))|$$

$$= |E \left( \frac{\partial g}{\partial x} \left( X(t + \Delta t) - X_{em}(t + \Delta t) \right) + O \left( \left( X(t + \Delta t) - X_{em}(t + \Delta t) \right)^2 \right) \right)$$

$$+ \frac{\partial g}{\partial x} \left( \hat{M}(t + \Delta t, t) \Delta t \right)$$
\[ \frac{\partial}{\partial x} \left( \mathcal{M}(t \rightarrow t, t)(\sigma^t) \right) \]

\[ \left\{ \mathbb{E} \left( \frac{\partial}{\partial x} \left( \mathcal{X}(t \rightarrow t, t) \mathcal{X}(t \rightarrow t, t) \sigma^t \right) \right) + O\left( \sigma^t \right) \right\} \]

\[ = \sim \sim \sim \sim = O\left( \sigma^t \right) \]

Gap in proof will be fixed in next lecture.

It turns out that the Euler-Marayama method can be shown to be first order accurate in a weak sense but we are missing a step in the argument. Basic idea is that the terms that spoil the first order strong accuracy are martingales that have mean zero.

Numerical stability: Short discussion in the paper by Higham, and in Kloeden and Platen Sec. 9.8.

Different notions of stability
Different notions of stability

Absolute stability: Kloeden and Platen Sec. 9.8:
\[ dX = \lambda X\, dt + \sigma X\, dW \]

The region of absolute stability in the complex plane are those values of \( \lambda \) for which the numerical method does not blow up when trying to solve this equation. Also probabilistic generalizations of more abstract stability criteria.

Funny things happen with stability for implicit methods with multiplicative noise.

Implicit Euler scheme for:

\[
X_{n+1} = X_n + \lambda X_{n+1}\, dt + \mu X_{n+1}\, dW_n
\]

\[
X_{n+1} = \frac{X_n}{(1 - \lambda \sigma - \sigma W_n)}
\]

How can one talk about stability of such a scheme when you have to worry about it blowing up for arbitrary choices of parameter \( \mu \). By the way this isn't probably even a consistent scheme -- need correction terms because noise has Ito interpretation. One strategy for developing higher order methods is actually to convert the equation to Stratonovich form because it makes the calculus easier.

Back to the question of a first-order strongly accurate method:

Milstein method:

\[
X_{n+1} = X_n + aX_n\, dt + bX_n\, dW_n
+ \frac{1}{2} bX_n\frac{\partial b}{\partial X} (X_n, t) (dW_n)^2 - \sigma^2 dt
\]

Strongly first order accurate.
Higher order methods are much more elaborate...

Also Milstein method is not straightforward for vector systems in general. The correction term then involves

\[
\int_t^s \left( W_i(s) - W_i(t) \right) dW_j(r)
\]

where \( W_i(s) \) and \( W_j(s) \) are independent Wiener processes driving two different components of the system. Not clear how to simulate this random variable; its not as easy as the case where \( i = j \). This is typical for higher order SDE methods: the higher order terms involve stochastic integrals (and iterated integrals) that have probability distributions which are hard to simulate. Kloeden and Platen Ch.5 develop methods to approximately simulate these kinds of random variables.

D. Talay: a different perspective. Use rather techniques like Romberg extrapolation, and things like stochastic Runge-Kutta 2 method that has good weak convergence properties.