No office hours Wednesday, March 28.

Class on Tuesday, March 27 moved to 2-3:50 PM in Sage 4203.

Special math seminar: Monday and Tuesday at 4 PM.

Existence and uniqueness of solutions to SDE's:

\[ dX = a(X, t)dt + b(X, t) dW(t) \]

\[ X(t=0) = X_0 \]

Mathematically, a solution of the SDE is defined to be a random function \( X(t) \) which satisfies the stochastic integral equation:

\[ X(t) - X(0) = \int_0^t a(X(t'), t') dt' + \int_0^t b(X(t'), t') dW(t') \]

**Theorem:** This SDE has a unique strong solution, which is adapted to the filtration \( \mathcal{F}_t^W \) and continuous in time, and depending continuously on the initial conditions and parameters provided:

1) \( a, b \) jointly measurable w.r.t.
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2) **Lipschitz condition**

\[
|a(x, t) - a(x', t)|, |b(x, t) - b(x', t)| \leq K|x - x'| \text{ deterministic}
\]

3) **growth bound**

\[
|xa(x, t)| \leq L^2 (1 + |x|^2)
\]

\[
|b(x, t)| \leq L|x| \text{ deterministic}
\]

**Idea of Proof:**

**Uniqueness:**

Consider two solutions \(x(t), \tilde{x}(t)\):

\[
\mathbb{E}\left(|x(t) - \tilde{x}(t)|^2\right)
\]

play in integral expressions on RHS of SDE, use Lipschitz condition

\[
x(t) = x(0) + \int_0^t b(x(s), s) \, ds + \int_0^t \sigma(x(s), s) \, dW(s)
\]

\[
\mathbb{E}\left(|x(t) - \tilde{x}(t)|^2\right) \leq 2 \int_0^t \mathbb{E}\left(|x(s) - \tilde{x}(s)|^2\right) \, ds
\]
\[ 0 \leq f(t) \leq g(t) + C \int_0^t e^{(t-s)} g(s) ds \]

**Intuitively:**
\[
\begin{align*}
\frac{df}{dt} &\leq C f + g' \\
f(0) &= g(0)
\end{align*}
\]

Bound \( f(t) \) by solving the differential equation.

Pick a time interval \([0, T]\)

Use Gronwall's inequality:
\[
\begin{align*}
f(t) &= |E(E(t))| \\
g(t) &= 0 \\
C &= 2K^2 [T+1]
\end{align*}
\]

\[ \Rightarrow |E(E(t))|^2 \geq 0 \text{ for } 0 \leq t \leq T \]

\[ \Rightarrow E(t) = 0 \text{ w/ prob. 1} \]

\[ \Rightarrow \mathbb{E}(X(t)) = \mathbb{E}(\hat{X}(t)) \text{ w/ prob. 1 [uniqueness]} \]
Existence:

Picard iteration (method of successive approximation)

\[
\begin{align*}
X^{(0)}(t) &= X_0 \\
X^{(n+1)}(t) &= X_0 + \int_0^t a(X^{(n)}(\tau), \tau) \, d\tau \\
&\quad + \int_0^t b(X^{(n)}(\tau), \tau) \, dW(\tau)
\end{align*}
\]

First check that the approximations all remain well-defined, don't blow up or run away.

\[(X(t) \in L^2)\]

Use growth bounds on coefficients:

\[
\begin{align*}
\mathbb{E}\left(\left|X^{(n+1)}(t)\right|^2\right) \\
\leq 3 \mathbb{E}\left|X_0^2 + 6L^2 [t+1] \int_0^t \mathbb{E}\left|X^{(n)}(\tau)\right|^2 d\tau\right|
\end{align*}
\]

\[
\Rightarrow \mathbb{E}\left(\left|X^{(n+1)}(t)\right|^2\right) \text{ has uniform bound for } t \in [0, T]
\]

and \[X^{(n+1)}(t) \in L^2\]
and $A(t) \leq L$

Now show successive approximations converge.

Use Lipschitz bounds:

$$|E\left(\left(\bar{X}^{(h+1)}(t) - \bar{X}^{(h)}(t)\right)^2\right)\right| \leq 2(T+1)K^2 \int_0^t dt' E\left(\left(\bar{X}^{(h)}(t') - \bar{X}^{(h-1)}(t')\right)^2\right)$$

for $0 \leq t \leq T$

Solve recursive $\gamma$:

$$|E\left(\left(\bar{X}^{(h)}(t) - \bar{X}^{(h-1)}(t)\right)^2\right)\right| \leq K^{-h} \int_0^t dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} \cdots \int_0^{t_1} dt' E\left(\left(\bar{X}^{(h)}(t') - \bar{X}^{(h-1)}(t')\right)^2\right)$$

$$|E\left(\left(\bar{X}^{(h)}(t) - \bar{X}^{(h-1)}(t)\right)^2\right)\right| \leq K^{-h} \int_0^t dt' \frac{(t-t')^{n-1}}{(n-1)!}$$

$$\times E\left(\left(\bar{X}^{(h)}(t') - \bar{X}^{(h-1)}(t')\right)^2\right)$$
\[ |E \left( \left( X^{(n)}(t) - X^{(1)}(t) \right)^2 \right) \leq C K^\alpha \] 

using growth bounds.

\[ \sup_{0 \leq t \leq T} \left| E \left( \left( X^{(n)}(t) - X^{(1)}(t) \right)^2 \right) \right| \leq C K^\alpha T^n \]

One can proceed to use inequality based on martingale property at the stochastic integral (Doob inequality).

\[ |E \left( \sup_{0 \leq t \leq T} \left( X^{(n+1)}(t) - X^{(1)}(t) \right)^2 \right) \leq C K^\alpha T^n \]

These imply convergence of successive approximates (in mean-square sense).

Define

\[ X(t) = X^{(1)} + \sum_{n=0}^{\infty} \left( X^{(n+1)}(t) - X^{(n)}(t) \right) \]
\[
X(t) = X_0 + \sum_{n=0}^{\infty} (X_n(t) - X_{n-1}(t)) \text{ converges in } \mathbb{F}
\]

Then one checks a few technical things to show that the function \(X(t)\) defined by this limit of successive approximations is indeed a solution of the SDE.

**Numerical methods**

Higham, "An Algorithmic Introduction to the Numerical simulation of Stochastic Differential Equations": excellent overview with MATLAB codes

Very good detailed textbook on higher order methods for solving SDEs:

Kloeden & Platen, *Numerical Solution of Stochastic Differential Equations*

See D. Tulyay for second-order stochastic Runge-Kutta and another point of view on what makes a good numerical method.

Let's assume SDE coefficients smooth.

\[
dX = a(X(t), t) \, dt + b(X(t), t) \, dW(t)
\]
This is a stochastic version of the forward Euler scheme for deterministic differential equations. But its validity and accuracy are not so...
This is a stochastic version of the forward Euler scheme for deterministic differential equations. But its validity and accuracy are not so straightforward.

In particular, this method wouldn't even be consistent if the SDE were interpreted in a Stratonovich sense. But since we are taking the Ito interpretation, which is related to coefficients being non-anticipating functions of the underlying randomness, it turns out that this Euler scheme is consistent for Ito SDE's. But we'll see that the scheme isn't really first order accurate. This numerical scheme is actually called the Euler-Marayama method because Marayama experimented with it in the 1950's. The numerical analysis of this and higher order schemes was developed later starting in the 1970s and 1980s.

Orders of accuracy:

Strong order of accuracy

Fix a time interval, $[0, T]$.

$X(x, t)$ is the numerical approximation

$\mathbb{E}(|X(T) - X(x, T)|) \leq C(T, 0, t)\delta$

⇒ numerical method has a strong order of accuracy.

This is called strong accuracy because it demands that the numerical approximation is close to the true solution, realization by realization with reference to the underlying noise source.

Weak order of accuracy

For any smooth bounded $g$
\[ \Gamma \quad \ldots \quad \nu \quad \text{and fixed time } T \]

\[ |E g(X(T)) - E g(X^0(T))| \leq C_{ij} (\delta t) \]

\[ \Rightarrow \text{ numerical method has } \delta \text{ weak order of accuracy.} \]