Stratonovich stochastic integral
\[
\int_0^b f(W(t), t) \, dW(t)
\]
\[
= \lim_{N \to 0} \sum_{j=1}^{n} \left( \frac{1}{2} f(W(t_{j-1}), t_{j-1}) + \frac{1}{2} f(W(t_j), t_j) \right) \times (W(t_j) - W(t_{j-1}))
\]
(like trapezoidal rule)

Midpoint rule gives same result,
but less convenient to implement

More general random function \( f(t, w) \)
which is piecewise continuous:

\[
\int_0^b f(t, w) \, dW(t)
\]
\[
= \lim_{N \to 0} \sum_{j=1}^{n} \frac{1}{2} \left( f(t_{j-1}, w) + f(t_j, w) \right) \times (W(t_j) - W(t_{j-1}))
\]

- not sure how well developed the theory is for this general kind of Stratonovich stochastic integral

For the case \( f(t, w) = f(t, W(t)) \)
then a detailed operational stochastic calculus is available. One way it can be developed is to relate it to the Ito stochastic calculus, then use the Ito rules.

Formally:
\[
f(t, W(t)) \, dW(t) = W(t) \, dw
\]
This is now of Ito form since the function multiplying the differential is evaluated at the beginning of the time interval \( dt \) in question. So we can use Ito stochastic rules to simplify this expression.

\[
\begin{align*}
&f(t, W(t)) \circ dW(t) = W(t) + dW \\
&= \frac{1}{2} \left( f(t, W(t)) + f(t + dt, W(t + dt)) \right) dW(t) \\
&\quad \uparrow \\
&\text{looks into future, not in Ito form}
\end{align*}
\]

\[
= \frac{1}{2} \left[ f(t, W(t)) + f(t, W(t)) \\
+ \frac{\partial f}{\partial t} (t, W(t)) \ dt + \frac{\partial f}{\partial x} (t, W(t)) \ dW \\
+ O(dt^3) + O(dW^2) + O(dW dt) \right] dW
\]

This is now of Ito form since the function multiplying the differential is evaluated at the beginning of the time interval \( dt \) in question. So we can use Ito stochastic rules to simplify this expression.

\[
= \frac{1}{2} \left[ 2f(t, W(t)) \ dW + O(dt \ dW) \\
+ \frac{\partial f}{\partial t} (t, W(t)) \ dW \\
+ O(c(dW^3) + O(dW^2) + O(dW^2 \ dt) \right]
\]

\[
f(t, W(t)) \circ dW(t) = f(t, W(t)) \ dW(t) \\
+ \left[ \frac{1}{2} \frac{\partial f}{\partial x} (t, W(t)) \ dt \right] \\
\text{Stratonovich- to- Ito- correction}
\]

More general transformation from Stratonovich to Ito stochastic differentials

\[
\int f \ dx = a(Y(t), t) \ dt + b(Y(t), t) \ dW(t)
\]

\[
h(t, Y(t)) \circ dW(t)
\]
Relative advantages/disadvantages of Ito/Stratonovich stochastic integrals

<table>
<thead>
<tr>
<th></th>
<th>Ito</th>
<th>Stratonovich</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chain rule</td>
<td>modified (Ito lemma)</td>
<td>standard (good for differential geometry)</td>
</tr>
<tr>
<td>Martingale properties</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>[ E \int_a^b f(t,u) , dW(t) = 0 ]</td>
<td>[ E \int_a^b f(t,u) , dW(t) ]</td>
<td></td>
</tr>
<tr>
<td>Variance:</td>
<td>[ E \left( \int_a^b f(t,u) , dW(t) \right)^2 ]</td>
<td>No good formula for variance</td>
</tr>
<tr>
<td>Generalization</td>
<td>Lebesgue integration theory</td>
<td>&gt;?</td>
</tr>
</tbody>
</table>

To summarize, in practice, to calculate the statistics of a Stratonovich stochastic integral, one usually has to use the Stratonovich-to-Ito transformation to convert to a Ito stochastic integral and use Ito stochastic calculus rules. This suggests that in many cases, the Ito stochastic integral is easier to compute with, but the exception is in cases where the chain rule has to be used a lot (differential geometry) then maybe Stratonovich calculus is worth working with because it keeps the chain rule simpler.

But when one is writing down a stochastic differential equation for model for a real system, should it be interpreted as an Ito SDE or a Stratonovich SDE or...?
SDE or a Stratonovich SDE or...?

\[ \begin{align*}
    \text{Itô} & \quad d\overline{X} = a(X(t),t) \, dt + b(X(t), t) \, dW(t) \\
    \text{Strat} & \quad d\overline{X} = a(X(t), t) \, dt + b(X(t), t) \, dW(t)
\end{align*} \]

Sometimes you don't have to worry because it doesn't matter:

\[ b(X(t), t) = b(t) \]

What if you have to worry about Itô vs. Stratonovich?

To decide, think about two time scales which will both be short if one is working with an SDE model

- \( \tau_c \) = correlation time of noise
- \( \tau_a \) = adjustment time for the system

\[ \tau_c, \tau_a \ll 1 \]

- \( \tau_c \ll \tau_a \ll 1 \): Itô SDE
  - finance, biology
- \( \tau_a \ll \tau_c \ll 1 \): Stratonovich SDE
  - engineering, physics

Model: G. Pavliotis \& A. Stuart

Inertial particle in turbulent flow

\[ \begin{align*}
    m \frac{d\overline{V}}{dt} &= -\delta \left( \overline{V} - \overline{V}(X(t), t) \right) \\
    \overline{U} &\quad \text{fluid velocity}
\end{align*} \]
\( \mathbf{U} \) denotes fluid velocity

\( \mathbf{v} \) : particle velocity

\( \mathbf{x} \) : particle position

\( \mathbf{U} \) is generated by an Itô SDE

\[
\frac{d \mathbf{U}}{dt} = A(\mathbf{U}) \, dt + B \, dW
\]

Let \( \mathbf{U} \) have correlation time \( T_c \)

Particle has adjustment time \( T_a = \frac{m}{\delta} \)

If one takes \( T_q, T_c \to 0 \), what kind of course-grained eqn for \( \mathbf{x} \) results?

\( T_q, T_c \to 0, \ \frac{T_c}{T_a} \to 0 \)

\[
\frac{d \mathbf{x}}{dt} = a(\mathbf{x}(t), t) \, dt + b(\mathbf{x}(t), t) \, dW(t)
\]

\( \text{Itô} \)

\( T_q, T_c \to 0, \ \frac{T_a}{T_c} \to 0 \)

\[
\frac{d \mathbf{x}}{dt} = a(\mathbf{x}(t), t) \, dt + b(\mathbf{x}(t), t) \, dW(t)
\]

\( \text{Strat} \)

\( T_c, T_e \to 0, \ \frac{T_a}{T_c} \to \text{constant} \)
SDE example:

Stochastic population growth model

Deterministic: \[ \frac{dX}{dt} = aX \]
\[ X(t=0) = X_0 \]

\( a \): growth rate

Stochastic: \[ \frac{dX}{dt} = \left( a + b \frac{dW}{dt} \right) X \]
\[ X(t=0) = X_0 \]

stochastic growth rate

which fluctuates w/ no memory

Write it as a formal SDE, we'll treat it as Itô:

\[ dX = a X dt + b X dW \]

One way to solve SDEs:

guess solution & plug in.

Our equation here is linear, so let's try to manipulate it to simpler form.
\( \frac{dX}{X} = a \, dt + b \, dW \)

- like sep. of variables

Is LHS = \( d \ln X \) ?

\[
d \ln X = U'(X) \, dX + \frac{1}{2} U''(X) (dX)^2
\]

\[
= \frac{1}{X} \, dX + \frac{1}{2} \left( -\frac{1}{X^2} \right) (dX)^2
\]

\[
dX^2 = b^2 \, X^2 \, dt
\]

\[
d \ln \bar{X} = \frac{dX}{X} - \frac{1}{2} b^2 \, \frac{X^2}{X^2} \, dt
\]

\[
= \frac{dX}{X} - \frac{1}{2} b^2 \, dt
\]

\[
\frac{dY}{X} = d \ln X + \frac{1}{2} b^2 \, dt
\]

\[
d \ln X + \frac{1}{2} b^2 \, dt = a \, dt + b \, dW
\]

\[
d \ln X = (a - \frac{1}{2} b^2) \, dt + b \, dW
\]

\[
\int_0^t d \ln X = \int_0^t (a - \frac{1}{2} b^2) \, dt + \int_0^t b \, dW
\]

\[
\ln X(t) - \ln X(0) = (a - \frac{1}{2} b^2) \cdot t + b \, W(t)
\]
\[ X(t) = X(0) \ e^{(a - \frac{1}{2} b^2) t + b \ W(t)} \]