Instructions. You are allowed to use one 8 1/2 x 11 inch sheet of paper of notes. No electronic equipment is allowed (this includes calculators, PDAs, computers, books, cellular phones). Do not collaborate in any way. In order to receive credit, your answers must be clear, legible, and coherent.

1. (3 points each) Fill in the blank with the letter corresponding to the best description. Use
   
   SHM = simple harmonic motion; OD = overdamped; CD = critically damped; UD = underdamped; R = resonant; B = beating; TSS = transient plus steady-state; E = exponential growth

   i) \[ \text{SHM} \ddot{u} + 4u = 0 \]
   \[ u(t) = a \cos 2t + b \sin 2t \]

   ii) \[ \text{B} \ddot{u} + 4u = \cos t \]

   iii) \[ \text{OD} \ddot{u} + 4u = \cos 2t \]

   iv) \[ \text{UD} \ddot{u} + \dot{u} + u = 0 \]
   \[ u = e^{rt} \Rightarrow r^2 + r + 1 = 0 \Rightarrow r = -\frac{1 \pm \sqrt{1 - 4}}{2} \]

   v) \[ \text{TSS} \ddot{u} + \dot{u} + u = \cos t \]

   vi) \[ \text{TSS} \ddot{u} + \dot{u} + u = \cos 2t \]

   vii) \[ \text{E} \ddot{u} - 4u = 0 \]
   \[ u = e^{rt} \Rightarrow r^2 - 4 = 0 \Rightarrow u(t) = a e^{2t} + b e^{-2t} \]

   viii) \[ \text{CD} \ddot{u} + 2\dot{u} + u = 0 \]
   \[ u = e^{rt} \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow r = -1 \pm \frac{\sqrt{4 - 4}}{2} = -1 \]

   ix) \[ \text{UD} \ddot{u} + \frac{1}{2} \dot{u} + u = 0 \]
   \[ u = e^{rt} \Rightarrow r^2 + \frac{1}{2}r + 1 = 0 \Rightarrow r = -\frac{1}{2} \pm \frac{\sqrt{4 - 4}}{2} \]

   x) \[ \text{SHM} \]

   xi) \[ \text{CD} \]

   xii) \[ \text{UD} \]

   xiii) \[ \text{TSS} \]

   xiv) \[ \text{B} \]

   xv) \[ \text{R} \]
2. (15 points) Solve the initial value problem \( y'' + ay = b, \ y(0) = 0, \ y'(0) = 0, \) where \( a > 0 \) and \( b \) are nonzero constants.

\[ \text{Step 1: Solve } y'' + ay = 0. \ \text{Try } y = e^{rt} \Rightarrow r^2 + a = 0 \Rightarrow r = \pm i \sqrt{a}. \]

\[ y(t) = A \cos \sqrt{a} t + B \sin \sqrt{a} t \]

\[ \text{Step 2: Find a particular solution by guessing } y = c. \ \text{Plug } \Rightarrow a c = b \]

\[ \Rightarrow y(t) = \frac{b}{a} \]

\[ \text{Step 3: General solution is } y(t) = A \cos \sqrt{a} t + B \sin \sqrt{a} t + \frac{b}{a} \]

\[ 0 = y(0) = A + \frac{b}{a} \Rightarrow A = -\frac{b}{a} \]

\[ 0 = y'(0) = [-A \sqrt{a} \sin \sqrt{a} t + B \sqrt{a} \cos \sqrt{a} t]_{t=0} = \sqrt{a} \Rightarrow B = 0 \]

\[ y(t) = -\frac{b}{a} \cos \sqrt{a} t + \frac{b}{a} \]

3. (10 points) Find the general solution of \( y' + ay = b, \) where \( a \) and \( b \) are nonzero constants.

The method above works here too! But we learned the method of integrating factors.

\[ e^{at} [y' + ay = b] \Rightarrow \frac{d}{dt} [e^{at} y] = b e^{at} \]

\[ \Rightarrow e^{at} y = b \int e^{at} dt + c = \frac{b}{a} e^{at} + c \]

\[ y(t) = \frac{b}{a} + c e^{-at} \]

2
4. (10 points) For \( y' = (y - 1)(y - 2)(y - 3) \):
   a) Find the equilibrium solutions and classify them according to their stability.

   \[
   \begin{align*}
   y = 1 & \text{ is unstable critical point} \\
   y = 2 & \text{ stable} \\
   y = 3 & \text{ unstable}
   \end{align*}
   \]

   b) Sketch the solution satisfying \( y(0) = 5/2 \).

5. (5 points) Find the general solution of \( y' + \frac{\sin x}{\cos y} = 0 \). You may leave your solution in implicit form.

   \[
   \frac{dy}{dx} = -\frac{\sin x}{\cos y} \quad \Rightarrow \quad \cos y \, dy = -\sin x \, dx
   \]

   \[
   \int \cos y \, dy = -\int \sin x \, dx + C
   \]

   solution is \( \sin y + \cos x = C \)
6. (15 points) A tank initially contains 120 liters of pure water. A mixture containing a concentration of \( \gamma \) g/liter of salt enters the tank at a rate of 2 liters/min, and the well-stirred mixture leaves the tank at the same rate.

a) Set up, but do not solve, the differential equation appropriate for finding the amount of salt in the tank at time \( t \). Specify the meaning of your dependent variable.

\[
Q(t) = \text{grams of salt in tank} \\
\text{(description in words of the meaning of } Q(t))
\]

\[
\frac{dQ}{dt} = (\text{rate salt flows in}) - (\text{rate salt flows out})
\]

\[
= \left( \frac{\text{salt flow}}{\text{concentration}} \right) \left( \frac{\text{volume of flow}}{\text{flowing in}} \right) - \left( \frac{\text{salt outflow}}{\text{concentration}} \right) \left( \frac{\text{volume of flow}}{\text{flowing out}} \right)
\]

\[
= \left( \frac{\gamma \text{g/l}}{2 \text{l/min}} \right) - \left( \frac{Q(t)}{120 \text{g/l}} \right) \left( \frac{2 \text{l/min}}{120 \text{l}} \right)
\]

\[
\frac{dQ}{dt} = 2\gamma - \frac{Q}{60}
\]

b) If the mixture flows in at 2 liters/min but flows out at 3 liters/min, find the volume of fluid in the tank at time \( t \).

\[ \text{Volume decreases by 1 l every minute} \quad \frac{dV}{dt} = -1 \]

\[ V(t) = 120 - t \]

c) Set up, but do not solve, the differential equation for finding the amount of salt in the tank under assumption b).

\[
\frac{dQ}{dt} = 2\gamma - \left( \frac{Q}{120-t} \right) \left( 3 \text{l/min} \right) = 2\gamma - \frac{3Q}{120-t}
\]