Ex 1: \[ f(x, y) = \begin{cases} \frac{x^3}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \]

\( (x, y) \neq (0, 0) \)

a) Show that \( f \) is continuous at the origin.

In polar coordinates: \( f(r\cos \theta, r\sin \theta) = \frac{r^3 \cos^3 \theta}{r^2} = r\cos \theta \rightarrow 0 \) as \( r \rightarrow 0 \)

Therefore \( f \) is continuous at the origin.

b) Find the partial derivatives of \( f \) away from the origin, we have:

\[ f_x = \frac{x^4 + 3x^2y^2}{r^2} = \cos^4 \theta + 3 \sin^2 \theta \cos^2 \theta \]
\[ f_y = \frac{-2y x^3}{(x^2 + y^2)^2} = -2 \sin \theta \cos \theta \]

But at \((0, 0)\):

\[ f_x(0, 0) = \lim_{x \to 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \to 0} \frac{x^4}{x^2} = 1 \]
\[ f_y(0, 0) = \lim_{y \to 0} \frac{f(0, y) - f(0, 0)}{y} = 0 \]

so \( \nabla f(0, 0) = (1, 0) \)

so \( f_x, f_y \) have no limit at \((0, 0)\) so \( f \) is not of class \( C^1 \) at \((0, 0)\)

c) Show that the directional derivative exists for every unit vector \( \mathbf{u} \):

\[ \frac{f(t\cos \theta, t\sin \theta) - f(0, 0)}{t} = \frac{t^4 \cos^3 \theta}{t^2} = \cos^3 \theta \]

So \( \nabla f(0, 0) \) is not \( C^1 \).

d) Show that, in spite of c), \( f \) is not differentiable at \((0, 0)\):

\[ \frac{f(r\cos \theta, r\sin \theta) - f(0, 0)}{r} = \frac{3}{r} \cos^3 \theta \cos \theta \]

Therefore \( f \) is not differentiable at \((0, 0)\).
Exercise 2: \[ f(x, y) = \begin{cases} \frac{\sqrt{x}}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \]

\[ f(x) = \frac{y \sqrt{x^6 + 4x^2y^6 - 9x^4y^4 - y^8}}{(x^2 + y^2)^3} \]

\[ f_x = \frac{x^4 + 4x^2y^6 - 9x^4y^4 - y^8}{(x^2 + y^2)^3} \]

Using polar coordinates, \( f_x, f_y \to 0 \) as \( r \to 0 \), implying that \( f_x, f_y \) are continuous at the origin.

Moreover, \( f_x(0, 0) = -1 \), \( f_x(x, 0) = x \).

Therefore, \( f_{xy}(0, 0) = \lim_{r \to 0} \frac{f_x(r \cos \theta, r \sin \theta) - f_x(0, 0)}{r} = -1 \).

Also, \( f_{yx}(0, 0) = \lim_{r \to 0} \frac{f_y(r \cos \theta, r \sin \theta) - f_y(0, 0)}{r} = 1 \).

Thus, \( f_{xy}(0, 0) = -1 \) and \( f_{yx}(0, 0) = 1 \).
Ex 3:

Let \( \xi = x + ct \), \( \eta = x - ct \), where \( c > 0 \) is a constant. Show that if \( \phi(\xi) \) and \( \psi(\eta) \) are of class \( C^2 \) on \( D \), then \( u = \phi(\xi) + \psi(\eta) \) is a solution of the wave equation

\[ c^2 u_{xx} = u_{tt} \]

Using the chain rule, we have

\[ u_x = \phi'(\xi) \xi_x + \psi'(\eta) \eta_x = \phi'(\xi) + \psi'(\eta) \]
\[ u_{xx} = \phi''(\xi) \xi_x + \psi''(\eta) \eta_x \]
\[ c^2 u_{xx} = c^2 \phi''(\xi) + c^2 \psi''(\eta) \tag{1} \]

\[ u_t = \phi'(\xi) \xi_t + \psi'(\eta) \eta_t = \phi'(\xi) c - \psi'(\eta) c \]
\[ u_{tt} = \phi''(\xi) c^2 + \psi''(\eta) c^2 = c^2 \phi''(\xi) + c^2 \psi''(\eta) \tag{2} \]

The right-hand side of (1) and (2) are equal. Therefore

\[ c^2 u_{xx} = u_{tt} \]

Ex 4:

Find the equation of the tangent plane to the surface \( y = x^2 - y^2 \) at the point \( P_0 = (1, 1, 2) \).

\[ f(x, y, z) = x^2 - y^2 - z = 0 \]

\[ \nabla f(x, y) = (2x, 2y, -1) \Rightarrow \nabla f(P_0) = (2, 2, -1) \]

The equation of the tangent plane is given by the formula

\[ \nabla f(P_0) \cdot (P - P_0) = 0 \]

Therefore, in this case, we have

\[ 2(x - 1) + 2(y - 1) - (3 - 2) = 0 \]
\[ 2x + 2y - 3 = 2 \]
Ex 5: The temperature in a rectangular box is approximately given by
\[ T = (x-x^2)(2y-y^2)z \]
\[ 0 \leq x \leq 1, \quad 0 \leq y \leq 2, \quad 0 \leq z \leq 1 \]
If a bee is located at \( (\frac{1}{2}, 1, 1) = P_0 \), in which direction should it fly in order to cool off as rapidly as possible?

\[ \nabla T(P_0) = \left[ \left(1-2x\right)(2y-y^2)z, \left(2-2y\right)z, \left(x-x^2\right)(2y-y^2) \right] \bigg|_{P=P_0} \]
\[ = (0, 0, 3/4) = \frac{3}{4} \hat{k} \]
We know that \( \nabla T \) is the direction of maximum increase and \(-\nabla T\) is the direction of most rapid decrease.
Therefore it should fly in the direction of \(-\hat{k}\) from its location \( (\frac{1}{2}, 1, 1) \).

Ex 6: Find the Taylor expansion of \( f(x, y) = xy^2 \) at \( P = (1, 1) \).

\[ f(P) = f(P_0) + (x-1) \cdot \nabla f(P_0) + \frac{1}{2} \left( (x-1) \cdot \nabla^2 f(P_0) \right) \cdot (x-1) \]
\[ = 1 + (x-1)(2y) + (y-1)(y-1) + (y-1)^2 + (x-1)(y-1)^2 \]

Ex 7: Maximize the volume \( V = xyz \) of a rectangular box, open at the top and having a fixed lateral surface \( A \).

\[ A = xy + 2xz + 2yz = 7, \quad g = xy + 2xz + 2yz - A = 0 \]
Lagrange multiplier method: \( \nabla V = \lambda \nabla g \)
\[ \begin{align*}
    yz &= \lambda (y + 2y) \\
    xy &= \lambda (x + 2z) \\
    xz &= \lambda (2x + 2y)
\end{align*} \]
Solving the system we get:
\[ x = y = 2z \]
\[ y = (\frac{1}{2})^{1/2} \]
\[ x = (\frac{A}{3})^{1/2} \]

Maximum value of \( V \) is \( \frac{1}{2} (\frac{A}{3})^{3/2} \)
Ex 8: Find the maximum of \( f(x_1, \ldots, x_n) = (x_1 \cdots x_n)^{1/n} \)
subject to the constraint \( g(x_1, \ldots, x_n) = (x_1 + \cdots + x_n) - S = 0 \)
\( x_i > 0 \)
\( S > 0 \)

We should have
\[
\nabla f = \lambda \nabla g
\]

For the components, we have
\[
\left\{ \begin{array}{l}
\frac{1}{n} (x_1 \cdots x_n)^{\frac{n-1}{n}} (x_2 \cdots x_n) = \lambda \Rightarrow (x_1 \cdots x_n)^{\frac{1}{n}} = n \lambda x_1 \\
\vdots \\
\frac{1}{n} (x_1 \cdots x_n)^{\frac{n-1}{n}} (x_1 \cdots x_{n-1}) = \lambda \Rightarrow (x_1 \cdots x_n)^{\frac{1}{n}} = n \lambda x_n
\end{array} \right.
\]

Let \( M = (x_1 \cdots x_n)^{1/n} \)
\[
\begin{align*}
x_1 &= \frac{M}{\lambda} \\
\vdots \\
x_n &= \frac{M}{\lambda}
\end{align*}
\]
\[
\Rightarrow x_1 \cdots x_n = \frac{M}{\lambda} \Rightarrow \lambda = \frac{M}{S}
\]

\[
\frac{S}{n}, \ldots, \frac{S}{n}
\]
The maximum at these points is
\[
f\left(\frac{S}{n}, \ldots, \frac{S}{n}\right) = \left(\frac{S}{n} \cdots \frac{S}{n}\right)^{1/n} = \frac{S}{n}
\]

Therefore
\[
(x_1 \cdots x_n)^{1/n} \leq \frac{S}{n} = \frac{x_1 + \cdots + x_n}{n}
\]

Ex 9:

Find the maximum of \( f(x_1, \ldots, x_n) = \sum_{k=1}^{n} \frac{x_k}{y_k} \)
subject to the constraint \( \sum_{k=1}^{n} x_k = 1 \)

We should have
\[
\nabla f = \lambda \nabla g
\]
or
\[
\left( \begin{array}{l}
y_1 \\
y_n
\end{array} \right) = \lambda \left( \begin{array}{l}
2x_1 \\
2x_n
\end{array} \right) \Rightarrow \left\{ \begin{array}{l}
x_1 = \frac{y_1}{2 \lambda} \\
x_n = \frac{y_n}{2 \lambda}
\end{array} \right.
\]
\[
1 = \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} \frac{y_k}{2 \lambda} \Rightarrow (2 \lambda)^2 \sum_{k=1}^{n} y_k = \sum_{k=1}^{n} y_k
\]
\[
\Rightarrow 2 \lambda = \left(\frac{\sum_{k=1}^{n} y_k}{\sum_{k=1}^{n} y_k} \right)^{1/2}
\]

\[
\text{Max } f = \sum_{k=1}^{n} \frac{x_k}{y_k} = \sum_{k=1}^{n} \frac{y_k}{2 \lambda} = 2 \lambda \left( \sum_{k=1}^{n} y_k \right)^{1/2} = 2 \lambda = \frac{\sqrt{\sum_{k=1}^{n} y_k}}{2}
\]

Consequence
\[
\sum_{k=1}^{n} x_k y_k = \left( \sum_{k=1}^{n} y_k \right) \left( \sum_{k=1}^{n} y_k \right)^{-1/2} \sum_{k=1}^{n} y_k y_k = \left( \sum_{k=1}^{n} y_k \right)^{1/2} \left( \sum_{k=1}^{n} y_k \right)^{-1/2} = 2 \lambda = \frac{\sqrt{\sum_{k=1}^{n} y_k}}{2}
\]

\[
\Rightarrow \sum_{k=1}^{n} x_k y_k \leq \left( \sum_{k=1}^{n} y_k \right)^{1/2} \left( \sum_{k=1}^{n} y_k \right)^{-1/2} = 2 \lambda = \frac{\sqrt{\sum_{k=1}^{n} y_k}}{2}
\]

and
\[
\Rightarrow \left( \sum_{k=1}^{n} y_k \right)^{1/2} \left( \sum_{k=1}^{n} y_k \right)^{-1/2} = 2 \lambda = \frac{\sqrt{\sum_{k=1}^{n} y_k}}{2}
\]

\[
\Rightarrow \left( \sum_{k=1}^{n} y_k \right)^{1/2} \left( \sum_{k=1}^{n} y_k \right)^{-1/2} = 2 \lambda = \frac{\sqrt{\sum_{k=1}^{n} y_k}}{2}
\]