12.) Solve the initial value problem. Sketch the graph of the solution and describe the behavior of the solution for increasing t.

\[ y'' - 6y + 9y = 0 \text{ where } y(0) = 0 \text{ and } y'(0) = 2. \]

First we let \( y = e^{rt} \), so that \( y' = re^{rt} \) and \( y'' = r^2 e^{rt} \). Substituting these values in, we have:

\[ r^2 - 6r + 9 = 0 \]
\[ (r - 3)(r - 3) = 0 \]
\[ r_{1,2} = 3 \]

Since we have repeated roots, the solution is of the following form:

\[ y = c_1 e^{3t} + c_2 te^{3t} \]

Now we plug in our first initial value:

\[ y(0) = 0 = c_1 e^{0} + c_2(0)e^{0} \]

leaving

\[ c_1 = 0. \]

Now taking one derivative of our general solution and plugging in our value of \( c_1 \) gives us:

\[ y'(t) = c_2 t(3e^{3t}) + c_2 e^{3t} \]
\[ y'(0) = 2 = c_2 \]

Thus our final solution is:

\[ y = 2te^{3t} \]

If you copy and paste
\[ \text{plot}(2*t*exp(3*t), t=0..10); \]
into Maple, you will get the graph for this problem. Clearly, as \( t \to \infty \), \( y \to \infty \).

30.) \( x^2 y'' + xy' + (x^2 - 0.25)y = 0 \), \( x > 0 \), \( y_1 = x^{-\frac{1}{2}} \sin x \)

For this problem we will use the method of reduction of order. First, let \( y = vy_1 \). Thus we have

\[ y = vx^{-\frac{1}{2}} \sin x \]
\[ y' = v(x^{-\frac{1}{2}} \cos x - \frac{1}{2} x^{-\frac{3}{2}} \sin x) + (x^{-\frac{1}{2}} \sin x)v' \]
\[ y'' = v'' x^{-\frac{1}{2}} \sin x + 2v'(x^{-\frac{1}{2}} \cos x - \frac{1}{2} x^{-\frac{3}{2}} \sin x) + v(-x^{-\frac{1}{2}} \sin x - \frac{1}{2} x^{-\frac{3}{2}} \cos x + \frac{3}{4} x^{-\frac{5}{2}} \sin x - \frac{1}{2} x^{-\frac{7}{2}} \cos x) \]
Plugging this into our original equation, we have:
\[ x^2 (v'' x^{-\frac{3}{2}} \sin x + 2v' (x^{-\frac{3}{2}} \cos x - \frac{1}{2} x^{-\frac{3}{2}} \sin x) + v (-x^{-\frac{3}{2}} \sin x - \frac{1}{2} x^{-\frac{3}{2}} \cos x + \frac{3}{4} x^{-\frac{3}{2}} \sin x - \frac{1}{2} x^{-\frac{3}{2}} \cos x)) + x (v (x^{-\frac{3}{2}} \cos x - \frac{1}{2} x^{-\frac{3}{2}} \sin x) + (x^{-\frac{3}{2}} \sin x) v') + (x^2 - \frac{1}{4}) (v x^{-\frac{3}{2}} \sin x) = 0 \]

Simplifying and putting everything in groups with respective coefficients \( v, v' \) and \( v'' \), we have the following:
\[ v'' (x^{-\frac{3}{2}} \sin x) + v' (2x^{-\frac{3}{2}} \cos x - \frac{1}{2} x^{-\frac{3}{2}} \sin x + \frac{3}{4} x^{-\frac{3}{2}} \sin x) + v (-x^{-\frac{3}{2}} \sin x + \frac{3}{4} x^{-\frac{3}{2}} \sin x + \frac{3}{4} x^{-\frac{3}{2}} \sin x - \frac{1}{2} x^{-\frac{3}{2}} \sin x - \frac{1}{2} x^{-\frac{3}{2}} \sin x) = 0 \]

Simplifying further, we are left with:
\[ \Rightarrow v'' x^{-\frac{3}{2}} \sin x + 2v' x^{-\frac{3}{2}} \cos x = 0 \]

Using the substitution
\[ v' = u(x) \]
\[ v'' = u'(x) \]
we are left with
\[ \frac{du}{dx} x^{-\frac{3}{2}} \sin x + 2ux^{-\frac{3}{2}} \cos x = 0 \]
which we can then use separation of variables to integrate. The integration process is as follows:
\[ \frac{du}{dx} x^{-\frac{3}{2}} \sin x = -2ux^{-\frac{3}{2}} \cos x \]
\[ \int \frac{du}{u} = -2 \int \frac{\cos x}{\sin x} dx \]
\[ \ln |u| = -2 \ln |\sin x| + C \]
\[ u = C_0 (\sin x)^{-2} \]
where \( C_0 \) is just \( e^C \). Next we plug \( v' \) back in for \( u \), we have
\[ \frac{dv}{dx} = C_0 \frac{1}{\sin x^2} \]
which we can also separate and integrate, yielding
\[ v = C_0 \int \frac{1}{\sin x^2} dx \]
which, using the fact that \( \frac{1}{\sin x} \) is equivalent to \( csc x \), yields
\[ v = C_0 \int csc^2 x dx \]
\[ \Rightarrow v = C_0 (-\cot x). \]

Letting \(-C_0 = C_1\) we have
\[ v = C_1 \cot x. \]
Now using the fact that $y_2 = vy_1$, we get

$$y_2 = (C_1 \cot x)(x^{-\frac{1}{2}} \sin x).$$

Since $\cot x = \frac{\cos x}{\sin x}$, we have that

$$y_2 = C_1 x^{-\frac{3}{2}} \cos x.$$

Finally, throwing out the arbitrary constant, we are left with

$$y_2 = x^{-\frac{1}{2}} \cos x.$$

This is our final answer.

*Note: If you have any questions on any of these or the ungraded problems, feel free to email me at rogerl@rpi.edu