The following problems were selected for grading: Section 3.1 # 1

Find the general solution to the following equation:

\[ y'' + 2y' - 3y = 0 \]

This is a second order, linear, homogeneous ordinary differential equation with constant coefficients.

We guess the form of the solution, \( y = Ae^{rt} \). Substitute and solve for viable values of the parameter, \( r \).

This yields the following characteristic polynomial:

\[ r^2 + 2r - 3 = 0 \]

we can factor this polynomial to find two distinct, real roots.

\[ (r + 3)(r - 1) = 0 \]
\[ r = -3 \quad r = 1 \]

Plugging these into the assumed functional form of the solution and taking an arbitrary linear combination yields the general solution:

\[ y_{gen} = C_1 e^{-3t} + C_2 e^t \]
Find the solution of the following initial value problem. Sketch the graph of the solution and describe its behavior for increasing $t$.

\[ y'' + 4y' + 5y = 0 \quad y(0) = 1 \quad y'(0) = 0 \]

We can solve this equation using the same method as above. The characteristic polynomial becomes:

\[ r^2 + 4r + 5 = 0 \]

Using the quadratic formula, we find that this equation has two complex roots:

\[ r = \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2} \]
\[ r = -2 \pm i \]

Using Euler’s identity we know that we can write the solution out of the following fundamental solutions:

\[ y_{gen} = C_1 e^{-2t} \cos(t) + C_2 e^{-2t} \sin(t) \]

Plugging in the initial conditions gives us the following system of equations to solve for $C_1$ and $C_2$.

\[ y(0) = 1 = C_1 \]
\[ y'(0) = 0 = -2C_1 + C_2 \]

Solving these equations simultaneously gives: $C_1 = 1$, $C_2 = 2$.

This gives the particular solution:

\[ y = e^{-2t} \cos(t) + 2e^{-2t} \sin(t) \]

This solution will display oscillations which decay exponentially to zero.

You can graph this solution accurately in Maple through the use of the following single line command:

\[ \text{plot}(\exp(-2*t) \times \cos(t) + 2 \times \exp(-2*t) \times \sin(t), t = 0..10); \]

(This command should work fine in older versions of Maple. With the dynamic formatting in Maple 10, I do not know whether it will work from a simply copy-paste operation)
The following problem appeared to engender the most difficulty across all of my sections, so I’ve included a detailed solution to help you along:

Section 3.2 # 22

Solve the initial value problem and determine the value of $\beta$ so that the solution decays to zero as $t \to \infty$.

$$4y'' - y = 0, \quad y(0) = 2 \quad y'(0) = \beta$$

Assume $y = Ae^{rt}$ and obtain the following characteristic polynomial:

$$4r^2 - 1 = 0$$

Solve to obtain: $r = \frac{1}{2}, \quad -\frac{1}{2}$. This yields the general solution:

$$y = Ae^{\frac{t}{2}} + Be^{-\frac{t}{2}}$$

Using the initial conditions we obtain the following system of equations involving $A, B$ and $\beta$.

$$2 = A + B$$

$$\beta = \frac{A}{2} - \frac{B}{2}$$

Now we solve these equation to get everything in terms of $\beta$. Using the first equation we find:

$$\beta = \frac{2 - B}{2} - \frac{B}{2}$$

$$-\beta + 1 = B$$

Plugging this into our first equation yields:

$$A = 1 + \beta$$

Now we write our solution in terms of these coefficients (which depend on $\beta$).

$$y = (1 + \beta)e^{\frac{t}{2}} + (1 - \beta)e^{-\frac{t}{2}}$$

In order for the solution to decay to zero, no part should grow as $t \to \infty$. Since the first function, $e^{\frac{t}{2}}$ would grow as $t$ gets large, we choose $\beta$ so as to zero this term.

This picks the unique value $\beta = -1$.

This concludes the problem.