Name:

MATP6640/DSES6770

Linear Programming

Spring 2006

Midterm Exam, Friday, April 7, 2006.

Please do all four problems. Show all work. No books or calculators allowed. You may use any result from class, the homeworks, or the texts, except where stated. You may use one sheet of handwritten notes. The exam lasts 110 minutes.

SOLUTIONS.

Q1	/30
Q2	/15
Q3	/15
Q4	/40
Total	/100

1. (30 points)
The following linear programming problem has an optimal solution x^* with x_1^*, x_2^*, and x_3^* basic:

$$
\begin{array}{lrcrcrcrcrcl}
\min & x_1 & + & 2x_2 & + & x_3 & + & x_4 & + & 5x_5 & & \\
\text{s.t.} & x_1 & + & x_2 & + & x_3 & + & 2x_4 & + & 2x_5 & = & b_1 \\
 & & & x_2 & + & x_3 & & & + & 3x_5 & = & b_2 \\
 & & & & & x_3 & + & x_4 & + & x_5 & = & b_3 \\
 & & & & & & & & & x_i & \geq & 0 \quad i = 1, \ldots, 5
\end{array}
$$

(a) (7 points) What is the dual linear program?

(b) (8 points) Use complementary slackness to solve the dual problem.

(c) (7 points) For this part, assume $x_1^* > 0$, $x_2^* > 0$, and $x_3^* > 0$. Can you conclude that there are multiple optimal solutions to the primal problem?

(d) (8 points) Assume b satisfies

$$-2b_1 + 2b_2 \mp b_3 = 0.$$

Can you conclude the optimal primal solution is degenerate?

(a) $\max\ b_1 y_1 + b_2 y_2 + b_3 y_3$

$$
\begin{array}{llllll}
\text{s.t.} & y_1 & & & \leq & 1 \\
 & y_1 & + y_2 & & \leq & 2 \\
 & y_1 & + y_2 & + y_3 & \leq & 1 \\
 & 2y_1 & & + y_3 & \leq & 1 \\
 & 2y_1 & + 3y_2 & + y_3 & \leq & 5
\end{array}
$$

(b) First three equations hold at equality,

So: $\boxed{y_1^* = 1,\ y_2^* = 1,\ y_3^* = -1}$

Check: dual feasible ✓

(c) Yes, since the fourth dual constraint holds at equality. Thus, the reduced cost for x_4 is equal to zero.

(d) Let $d = (-2, 2, -1)$, so $d^T b = 0$.
Then $A^T d \leq 0$, so there exist multiple optimal dual solutions, of the form $y^* + \lambda d$, $\lambda \geq 0$.
Thus, the primal optimal solution must be degenerate.

2. (15 points)
   Consider the LP:

$$\begin{array}{lrcrcrcl} \min & 2x_1 & + & 6x_2 & + & 3x_3 & & \\ \text{s.t.} & x_1 & + & x_2 & + & x_3 & = & 6 \\ & x_1 & - & 2x_2 & + & 4x_3 & = & 9 \\ & & & & & x_i & \geq & 0 \quad i = 1, \ldots, 3 \end{array}$$

   Show that $x = (3, 1, 2)$ is on the central trajectory for this problem.

Need $x_i s_i = \mu$ for all i, for some $\mu > 0$.

So $s_1 = \frac{\mu}{3}$, $s_2 = \frac{\mu}{1}$, $s_3 = \frac{\mu}{2}$

Also, $A^T y + s = c$, so $s = c - A^T y$.

Thus:

$$\frac{\mu}{3} = 2 - y_1 - y_2$$
$$\mu = 6 \cancel{+} - y_1 + 2y_2$$
$$\frac{\mu}{2} = 3 - y_1 - 4y_2$$

Solution: $\boxed{\mu = 6, \quad y_1 = y_2 = 0}$

3. (15 points)
Given n-vectors $x > 0$ and $s > 0$, show that the value of the scalar η that minimizes $||XSe - \eta e||^2$ is $\eta = \frac{x^T s}{n}$. Here X and S denote diagonal matrices containing the entries in x and s respectively, and e is the n-vector of ones.

$$f(\eta) = ||XSe - \eta e||^2$$

$$= \sum_{i=1}^{n} (x_i s_i - \eta)^2$$

$$\frac{df}{d\eta} = -2 \sum_{i=1}^{n} (x_i s_i - \eta) = 2(n\eta - x^T s)$$

So $\frac{df}{d\eta} = 0$ when $\boxed{\eta = \frac{x^T s}{n}}$

4. (40 points, each part is worth 20 points)

The following is a primal-dual pair of linear programs:

$$\begin{array}{llcll} \min & c^T x & & & \\ \text{s.t.} & Ax & = & b & (P) \\ & Hx & = & h & \\ & x & \geq & 0 & \end{array} \qquad \begin{array}{llcccccl} \max & b^T y & + & h^T u & & & \\ \text{s.t.} & A^T y & + & H^T u & \leq & c & (D) \end{array}$$

Assume (P) and (D) are both feasible. Let z denote the optimal value of (P) and (D). For any appropriately dimensioned vector π, define

$$\Theta(\pi) := \min\{c^T x + \pi^T (h - Hx) : Ax = b, x \geq 0\}$$

This value $\Theta(\pi)$ provides an underestimate for z. The best lower bound is obtained by maximizing $\Theta(\pi)$.

(a) Show that the maximum value of $\Theta(\pi)$ is equal to z.

(b) The maximum value of $\Theta(\pi)$ can be overestimated by solving a linear programming problem of the form

$$\begin{array}{lcccccl} \max & & & \theta & & & \\ \text{s.t.} & F\pi & + & d\theta & \leq & v & (LD) \end{array}$$

where F is a matrix and d and v are vectors. Assume an optimal solution $\bar{\pi}$, $\bar{\theta}$ for (LD) is known. Let $\bar{x}$ solve $\Theta(\bar{\pi})$, with corresponding dual multipliers $\bar{y}$. Assume $\bar{\theta} > \Theta(\bar{\pi})$. Give a valid linear constraint that must be satisfied by π and $\Theta(\pi)$.

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(a) $$\theta(\pi) = \min_x \; c^T x + \pi^T (h - Hx)$$
$$\text{s.t.} \quad Ax = b$$
$$x \geq 0$$

Dual: $$\max_y \; b^T y + \pi^T h$$
$$\text{s.t.} \quad A^T y \leq c - H^T \pi$$

So $$\theta(\pi) = \begin{cases} -\infty & \text{if } \pi \text{ doesn't satisfy } A^T y + H^T \pi \leq c \text{ for some } y. \\ \max_y \; b^T y + \pi^T h \;\; \text{s.t. } A^T y \leq c - H^T \pi & \text{otherwise.} \end{cases}$$

Thus, $$\max \theta(\pi) = \max_\pi \max_y \; b^T y + \pi^T h$$
$$\text{s.t.} \quad A^T y \leq c - H^T \pi$$

This is exactly the LP dual problem.

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(b) $$\theta(\pi) = \cancel{b} h^T\pi + \min_{\substack{\text{s.t.}\ Ax=b \\ x \geq 0}} (c - H^T\pi)^T x$$

$$\leq h^T\pi + (c - H^T\pi)^T \bar{x}$$

If all cuts of this form:

Get
$$\max \ \theta$$
$$\text{s.t.} \ (Hx^k - h)^T\pi + \theta \leq c^T x^k \quad \forall \text{ extreme points } x^k \text{ of } Q$$

Dual problem:

$$\min \ \sum (c^T x^k)\lambda_k$$
$$\text{s.t.} \ \sum (Hx^k - h)\lambda_k = 0$$
$$\sum \lambda_k = 1$$
$$\lambda_k \geq 0 \quad \forall k$$

Equivalently:

$$\min \ \sum (c^T x^k)\lambda_k$$
$$\text{s.t.} \ H\left(\sum \lambda_k x^k\right) = h$$
$$\sum \lambda_k = 1$$
$$\lambda_k \geq 0$$

This is exactly the Dantzig-Wolfe Master Problem.