Circle your Section Number:

1 (8 a.m., Stathos)  2 (9 a.m., Soliman)
3 (1 p.m., Stathos)  4 (3 p.m., Soliman)

Name ____________________________

SOLNS

Instructions.

Please work all four problems.
Please show all work clearly and in reasonable detail. Answers without appropriate supporting work or requested explanations may not receive full credit.
Use of books, notes, or calculators is not permitted.
1 (a) (20pts) Find the critical point for the function

\[ f(x, y) = x^2 - 3xy - y^2 - 5x + 14y + 3, \]

and show that it is a saddle point.

\[
\begin{aligned}
    f(x, y) &= x^2 - 3xy - y^2 - 5x + 14y + 3 \\
    f_x &= 2x - 3y - 5 = 0 \\
    f_y &= -3x - 2y + 14 = 0
\end{aligned}
\]

Set up the system of equations:

\[
\begin{aligned}
    3f_x + 2f_y &= -9y - 15 - 4y + 26 = 0 \\
    -13y + 13 &= 0 \quad \Rightarrow \quad y = 1
\end{aligned}
\]

Hence \( 2x - 3 - 5 = 0 \Rightarrow 2x = 8 \Rightarrow x = 4 \)

Critical point \((4, 1)\)

Check the type of critical point:

\[
d = \begin{vmatrix}
    f_{xx} & f_{xy} \\
    f_{yx} & f_{yy}
\end{vmatrix}
\]

\[
\begin{aligned}
    f_{xx} &= 2, \quad f_{yy} = -2, \quad f_{xy} = f_{yx} = -3 \\
    d &= \begin{vmatrix}
        2 & -3 \\
        -3 & -2
    \end{vmatrix} = -4 - 9 = -13 < 0
\end{aligned}
\]

Hence the critical point is a saddle point.
1. (b) (5pts) Which one of the following three plots represents the surface of part (a)?
1. (c) (5pts) Write an equation for the tangent plane to the surface 
\[ z = x^2 - 3xy - y^2 - 5x + 14y + 3 \] at \((4,1,0)\).

\[ \nabla f = \begin{bmatrix} x^2 - 3xy - y^2 - 5x + 14y + 3 \n \end{bmatrix} \]
\[ F_x = 2x - 3y - 5, \quad F_y = -3x - 2y + 14, \quad F_z = -1 \]
\[ F_x(4,1,0) = 8 - 3 - 5 = 0, \quad F_y = -12 - 2 + 14 = 0 \]

Hence, the equation for the tangent plane at \((4,1,0)\) is 
\[ 0(x - 4) + 0(y - 1) - (z - 0) = 0 \]

or \[ z = 0 \]
2. The electrical potential $V$ at a point $(x,y,z)$ over a certain region of space is given by

$$V(x,y,z) = 2x^2 - 3xy - y^2 - z^2.$$ 

(a) (10pts) The electric field is given by $E = -\nabla V$. Find the electric field at every point $(x,y,z)$.

$$E = -\nabla V = -\frac{\partial V}{\partial x} \hat{i} - \frac{\partial V}{\partial y} \hat{j} - \frac{\partial V}{\partial z} \hat{k} =$$

$$= -(4x - 3y) \hat{i} - (3x - 2z) \hat{j} + 2z \hat{k}$$

(b) (10pts) Find the rate of change of potential, that is, the **directional derivative** at the point $P(3,4,5)$ in the direction of the vector $v = i + 2j - 2k$. Is the potential increasing or decreasing in the direction of $v$?

$$D_v V = \nabla V \cdot v \quad \text{Here} \quad v = \frac{1}{||v||} \left[ \frac{i}{\sqrt{1+4+4}} \right] \frac{1}{\sqrt{17}} =$$

$$= \frac{1}{\sqrt{17}} (1 + 2 - 2) \quad \Rightarrow \quad \nabla V = (12 - 12) \hat{i} + (-9 - 5) \hat{j} - 10 \hat{k}$$

$$D_v V = \frac{1}{\sqrt{17}} (12 - 12) (-17) - 10 \hat{k} = -10$$

(c) (10pts) In what direction does $V$ decrease most rapidly at $P$? What is this maximum rate of decrease at $P$?

$$V \text{ decreases most rapidly in the direction } -\nabla V = \frac{17}{17} \hat{k} \text{ at } P. \quad \text{Maximum rate of decrease}$$

$$-||\nabla V|| = -\sqrt{17^2 + 10^2} = -17.89$$

(d) (5pts) Determine whether or not the potential distribution is in a state of **equilibrium** so that $V$ satisfies Laplace’s equation

$$V_{xx} + V_{yy} + V_{zz} = 0.$$ 

$$V_{xx} = 4, \quad V_{yy} = -2, \quad V_{zz} = -2$$

$$V_{xx} + V_{yy} + V_{zz} = 4 - 2 - 2 = 0$$

Yes. Potential is in equilibrium.
3. (15pts) Consider the function $w = e^{-f(x-t)}$.

Use the chain rule to show that $w_x + w_t + w = 0$.

Let $w = x+u$ and $w = e^{-f(u)}$.

\[
\begin{align*}
  w_x & = e^{-f(u)} \left( f'(u) \right) \\
  w_t & = -e^{-f(u)} + e^{-f(u)} u_t \\
       & = -e^{-f(u)} + e^{-f(u)} x(t) \\
\end{align*}
\]

\[
\begin{align*}
  w_x + w_t + w & = e^{-f(u)} - e^{-f(u)} - e^{-f(u)} + e^{-f(u)} \\
                  & = 0.
\end{align*}
\]
4. Given polar coordinates $x = r \cos \theta$ and $y = r \sin \theta$.
(a) (5pts) Show that
\[ \tan \theta = y/x \text{ and that } \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2} \]

\[ x = r \cos \theta, \quad y = r \sin \theta \Rightarrow \frac{y}{x} = \frac{r \cos \theta}{r \sin \theta} = \tan \theta \]
\[ \Rightarrow \theta = \tan^{-1} \left( \frac{y}{x} \right), \quad \frac{\partial \theta}{\partial y} = -\frac{1}{x^2} \left( \frac{y}{x} \right)^2 \left( \frac{y}{x} \right) = -\frac{y}{x^2 + y^2} \]

(b) (5pts) Evaluate $\partial \theta / \partial x$.
\[ \frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2} \]

(c) (10pts) Find
\[ r(x, y), \quad \frac{\partial r}{\partial x}, \text{ and } \frac{\partial r}{\partial y} \]
in terms of $x$ and $y$.
\[ x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2 (\sin^2 \theta + \cos^2 \theta) - r^2 \]
\[ r = \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2} \]
\[ \frac{\partial r}{\partial x} = \frac{1}{2} (x^2 + y^2)^{-1/2} \frac{\partial}{\partial x} (x^2 + y^2) = \frac{x}{\sqrt{x^2 + y^2}} \]
\[ \frac{\partial r}{\partial y} = \frac{1}{2} (x^2 + y^2)^{-1/2} \frac{\partial}{\partial y} (x^2 + y^2) = \frac{y}{\sqrt{x^2 + y^2}} \]