Remark/retraction:

Ito stochastic integral $Z(t) = \int_0^t f(t, \omega)\, dW(t, \omega)$

is not nec. a (strong) Markov process.

04/26/04 Connection between SDEs and diffusion processes

The ~~Ito~~ Ito SDE

$$d\vec{X}(t) = \vec{v}(\vec{X}(t), t)\,dt + \sigma(\vec{X}(t), t)\,d\vec{W}(t)$$

is ~~associated~~ is equivalent to a diffusion process with

drift $\vec{v}(\vec{x}, t)$

diffusion $K(\vec{x}, t) = \frac{1}{2}\sigma^{+}\sigma$

Detailed proof: Kloeden + Platen Sec. 4.6

By defn the solution to an Ito SDE is a solution of

$$\vec{X}(t) - \vec{X}(0) = \int_0^t \vec{v}(\vec{X}(s), s)\,ds + \underbrace{\int_0^t \sigma(\vec{X}(s), s)\,d\vec{W}(s)}_{\text{Ito integral}}$$

$\cancel{\mathbb{E}\left(\left(\frac{\vec{X}(t) - \vec{X}(s)}{t-s}\right)^2\right) = ?}$

$$\mathbb{E}\left(\frac{\vec{X}(t) - \vec{X}(s)}{t-s} \,\middle|\, \vec{X}(s) = \vec{x} \right)$$

$$= \mathbb{E} \int_s^t \vec{v}(\vec{X}(s'), s')\, ds'$$

$$= \int_s^t \mathbb{E}\, \vec{v}(\vec{X}(s'), s')\, ds'$$

$$\lim_{t \downarrow s} \mathbb{E}\left(\frac{\vec{X}(t) - \vec{X}(s)}{t-s} \,\middle|\, \vec{X}(s) = \vec{x} \right) = \lim_{t \downarrow s} \frac{\int_s^t \cancel{\mathbb{E}}\, \vec{v}(\vec{x}, s)\, ds}{t-s}$$

$$= \vec{v}(\vec{x}, s) \qquad \text{(drift)}$$

Diffusivity: (will just prove for scalar case)

$$\mathbb{E}\left(\frac{(X(t) - X(s))^2}{2(t-s)} \,\middle|\, X(s) = x \right)$$

$$= \frac{\cancel{1}}{2(t-s)} \mathbb{E}\left(\left(\int_s^t \vec{v}(X(s'), s')\, ds'\right)^2 + \left(\int_s^t \sigma(X(s'), s')\, dW(s')\right)^2 + 2\left(\int_s^t v(X(s'), s')\, ds'\right)\left(\int_s^t \sigma(X(s'), s')\, dW(s')\right) \right)$$

As $t \downarrow s$: First term is $\frac{1}{t-s} O((t-s)^2) \to 0$

Third term is $\frac{1}{t-s} O((t-s)^{3/2}) \to 0$

Second term:

$$\frac{1}{2(t-s)} \mathbb{E}\left(\left(\int_s^t \sigma(X(s'), s')\, dW(s')\right)^2\right)$$

$$= \frac{1}{2(t-s)} \int_s^t \mathbb{E}\left(\sigma(X(s'), s')\right)^2 ds'$$

by Ito stoch integral property

As $t \downarrow s$, this $\to \frac{1}{2(t-s)} \int_s^t (\sigma(x, s))^2 ds'$

$$= \frac{\sigma(x,s)^2 (t-s)}{2(t-s)}$$

$$K(x,s) = \frac{1}{2} \sigma(x,s)^2$$

How work with stochastic calculus in practice?

Like ordinary calculus, but the chain rule is different!

<u>Ito formula</u> (stochastic chain rule for "Ito" stochastic calculus)

Let $Y(t) = U(t, X(t))$

where U is a given function

$X(t)$ satisfies an <u>Ito</u> SDE

$$dX(t) = e(t,\omega)\,dt + f(t,\omega)\,dW(t)$$

Then $Y(t)$ satisfies SDE

$$dY(t) = \frac{\partial U(t, X(t))}{\partial t}\,dt + \frac{\partial U(t, X(t))}{\partial x}\,dX(t) + \frac{1}{2} f(t,\omega)^2 \frac{\partial^2 U}{\partial x^2}(t, X(t))\,dt$$

$$= \left(\frac{\partial U}{\partial t} + e\frac{\partial U}{\partial x} + \frac{1}{2} f^2 \frac{\partial^2 U}{\partial x^2}\right) dt + f\,dW(t)$$

Idea of origin of correction term:

Graphically:

Analytically: (Kloeden + Platen Ch. 3)

Consider the increment in $Y(t)$ over a short time Δt

Taylor expansion (stochastic)

$$\Delta Y(t) = U(t+\Delta t, X(t+\Delta t)) - U(t, X(t))$$

$$= \Delta t \frac{\partial U}{\partial t} + \Delta X(t) \frac{\partial U}{\partial x} + \frac{1}{2} (\Delta X(t))^2 \frac{\partial^2 U}{\partial x^2}$$

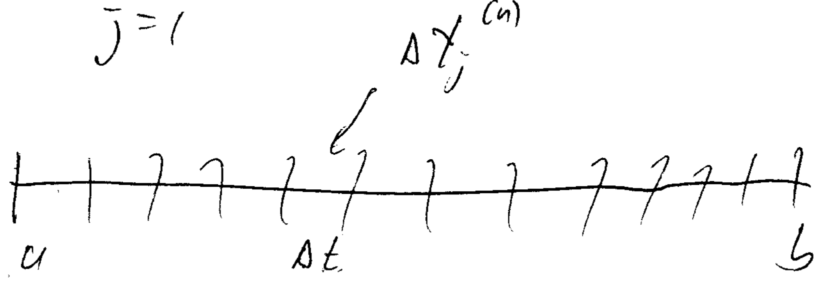
$$+ O(\Delta t \, \Delta X(t)) + O((\Delta t)^2) + O((\Delta X(t))^3)$$

where $\Delta X(t) = X(t+\Delta t) - X(t)$

One needs to keep track of all terms which are $\gtrsim \Delta t$

because over a finite time interval $[a,b]$

$$Y(b) - Y(a) = \sum_{j=1}^{n} \Delta Y_j^{(n)}$$



$$\Delta t = \frac{b-a}{n}.$$

So $n \cong \dfrac{b-a}{\Delta t}$

The tricky part is that

$$\Delta X(t) = e(t,\omega)\,\Delta t + f(t,\omega)\,\Delta W(t)$$

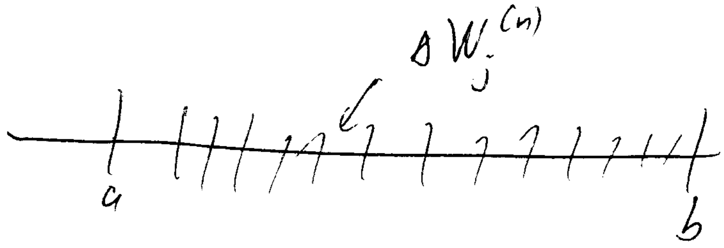
and $\mathbb{E}\left((\Delta W(t))^2\right) = \Delta t$

so $\Delta W(t)$ has stand. dev. $\sqrt{\Delta t}$

$$(\Delta X(t))^2 = f^2(t,\omega)\,(\Delta W(t))^2 + \underbrace{O((\Delta t)^2) + O(\Delta t\,\Delta W(t))}_{\text{can be neglected}}$$

$$\mathbb{E}\left((\Delta W(t))^2 \right) = \Delta t \qquad \text{but } \underbrace{(\Delta W(t))^2}_{\text{random}} \neq \underbrace{\Delta t}_{\text{deterministic}}$$

Recall $\displaystyle\lim_{n\to\infty} \sum_{j=1}^{n} \left(\Delta W_j^{(n)}\right)^2 = b - a$



So by law of large numbers,
the only relev effect of
$(\Delta W(t))^2$ over a finite time
interval is same as replacing
by mean $\mathbb{E}(\Delta W(t)^2) = \Delta t$

So $(\Delta X(t))^2$ can be replaced by

$$(\Delta X(t))^2 = f^2(t,\omega)\,\Delta t + O\left((\Delta t)^{3/2}\right)$$

$$\Delta Y(t) = \Delta t \frac{\partial U}{\partial t} + \Delta X(t) \frac{\partial U}{\partial x} + \frac{1}{2} f^2 \frac{\partial^2 U}{\partial x^2} \Delta t$$

$$+ O\left((\Delta t)^{3/2}\right)$$

Shorthand rule for manipulating Ito stochastic derivatives is to use ordinary Taylor expansion but remember dW is like $dt^{1/2}$

$dW(t)^2$ can be replaced by dt

$dW\,dt$ can be dropped, etc.

Corollary: ~~See~~ Ito Stochastic product rule:

If $dX(t) = e\,dt + f\,dW$
$dY(t) = g\,dt + h\,dW$

then

$$d(X(t)\,Y(t)) = Y\,dX + X\,dY + \underline{dX\,dY}$$

$$= (eY + gX + fh)\,dt + (fY + hX)\,dW(t)$$

Used Ito stochastic calculus rule to write $dX\,dY = fh\,dt$

How does this work for calculating integrals?

Consider $\int_0^t W(s)\,dW(s)$.

Guess that integral is $\frac{1}{2} W(t)^2$ or something like it.

Check:

$$d\left(\tfrac{1}{2} W(t)^2\right) = W(t)\,dW(t) + \tfrac{1}{2}\,\underline{dW(t)\,dW(t)}$$

$$= W(t)\,dW(t) + \tfrac{1}{2}\,dt$$

Integrating both sides:

$$\tfrac{1}{2} W(t)^2 - 0 = \int_0^t W(s)\,dW(s) + \tfrac{1}{2}\int_0^t ds$$

$$= \int_0^t W(s)\,dW(s) + \tfrac{1}{2} t$$

Itô: $\int_0^t W(s)\,dW(s) = \frac{1}{2} W(t)^2 - \frac{1}{2} t$

General method for integrating stochastically: guess antiderivative, differentiate it, and read off corrections

Other topics:

Existence + uniqueness for SDE's
by Picard iteration (K+P 4.5)

Stability + Lyapunov functions
$\downarrow$
Supermartingales

$$\mathbb{E}(V(t, X(t)) \mid \mathcal{Q}_s) \leq V(s, X(s))$$

for $t > s$

K+P Sec. 6.3

Stratonovich stochastic calculus

Think of discretizing $\qquad$ K+P Sec. 4.9

$$\int_a^b f(t,\omega) \underbrace{\circ\, dW(t,\omega)}_{\uparrow}$$

Stratonovich

- discretize by midpoint or trapezoidal rule

This gives different result than Ito.

Connection between Ito + Stratonovich:

$$\underline{h(t, X(t)) \circ dW(t)} = h(t, X(t))\, dW(t)$$

$$+ \frac{1}{2} \frac{\partial h}{\partial x}(t, X(t))\, \underline{\underline{dX(t)\, dW(t)}}$$

Ito	Stratonovich
Modified chain rule	Ordinary chain rule! - good for SDE on manifolds
Martingale	Not Martingale
$\mathbb{E} \int_s^t f(s', \omega)\, dW(s', \omega) = 0$	No analogue
$\mathbb{E} \left(\int_s^t f(s', \omega)\, dW(s') \right)^2 = \int_s^t \mathbb{E} f^2(s', \omega)\, ds$	

Which one to use in a model?

Let T_c = noise correlation time

T_a = dynamical adjustment time

Rough idea: when both T_c, T_a small

$T_c \ll T_a \Rightarrow$ Ito (finance, biology)

$T_c \gg T_a \Rightarrow$ Stratonovich (physics, engineering)

Rigorous example: G. Pavliotis + A. Stuart

Simple SDE example

$$\frac{dX}{dt} = \left(a + b\frac{dW}{dt}\right)X$$

or more formally

$$dX = aX\,dt + bX\,dW$$

Start w/ Ito interpretation:

Solve by trial and error

Let $Y = \ln X$

$$dY = \frac{dX}{X} - \frac{1}{2}\frac{(dX)^2}{X^2}$$

$$= \frac{dX}{X} - \frac{1}{2}\frac{b^2X^2\,dt}{X^2}$$

$$= \frac{dX}{X} - \frac{1}{2}b^2\,dt$$

$$= a\,dt + b\,dW - \frac{1}{2}b^2\,dt$$

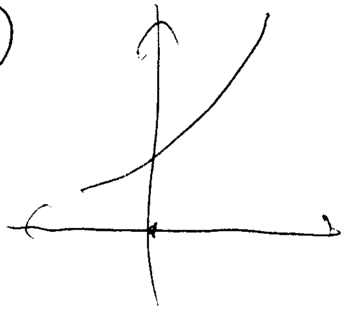
$$dY = \left(a - \frac{1}{2}b^2\right)dt + b\,dW$$

Integrate both sides:

$$Y(t) - Y(0) = (a - \tfrac{1}{2} b^2)t + b\, W(t)$$

Plugging in $X(t) = e^{Y(t)}$

$$X(t) = X(0)\, e^{(a - \frac{1}{2} b^2)t + b W(t)}$$



$$\mathbb{E}\, X(t) = X(0)\, e^{at}$$ by using

gen fn for Gaussian rv:

$$\mathbb{E}\, e^{b\, W(t)} = e^{\frac{1}{2} b^2 t}$$

So note $e^{-at} X(t)$ is a martingale.

Stratonovich: $dX = a X\, dt + b X \circ dW$

Ordinary calculus:

$$X(t) = X(0)\, e^{at + b W(t)}$$

$$\mathbb{E}\, X(t) = X(0)\, e^{(a + \frac{1}{2} b^2)t}$$

Numerical solution of SDE
- Higham article online

$$dX(t) = a(X,t)\,dt + b(X,t)\,dW(t)$$

Simplest scheme Euler-Maruyama

$$X_n = X(n\Delta t)$$

$$X_{n+1} = X_n + a(X_n, t_n)\Delta t + b(X_n, t_n)\Delta W_n$$

How accurate

½ order in <u>strong</u> sense!

order 1 in weak sense.

More accurate schemes: use stochastic Taylor expansions (K+P)

Milstein method (order 1 both strong + weak)

$$X_{n+1} = X_n + a(X_n, t_n)\Delta t + b(X_n, t_n)\Delta W_n + \frac{1}{2} b(X_n, t_n)\frac{\partial b}{\partial x}(X_n, t_n)\left(\Delta W_n^2 - \Delta t\right)$$