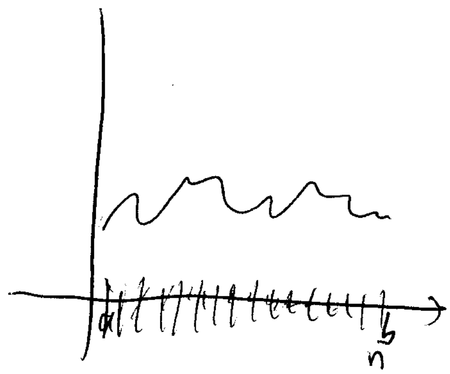
04/22/04 Mathematical Foundations of SDE's

The difficulty with $\vec{W}(t)$ is that it has unbounded variation.

$$\sup_{P} \sum_{j=1}^{n} |\vec{W}(t_j^{(n)}) - \vec{W}(t_{j-1}^{(n)})| = \infty$$

P: partitions over finite interval $[a,b]$

Kloeden + Platen Sec. 2.4



One step in proof of unbounded variation of Wiener process:

$$\lim_{\|P\|\to 0} \sum_{j=1}^{n} |\vec{W}(t_j^{(n)}) - \vec{W}(t_{j-1}^{(n)})|^2 = b-a$$

(deterministic) Law of Large Numbers

with prob. 1

take equally spaced partitions on $[a,b]$ with $n=2^m$ points. This fact is behind stochastic rule $(dW)^2 = dt$

For smooth functions limit would be 0.

On the other hand, $\vec{W}(t)$ is a martingale:

$$\mathbb{E}\,|\vec{W}(t)| < \infty$$

$$\mathbb{E}\left(\vec{W}(t) \mid \mathcal{a}_s\right) = \vec{W}(s) \quad \text{for } t > s.$$

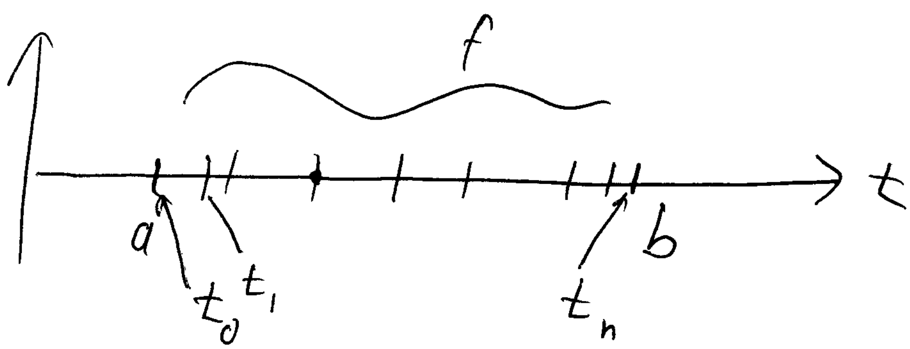
Martingales have rough but controllable fluctuations, so can integrate with them.

We will define the Ito ~~[illegible]~~ stochastic integral.

Riemann-Stieltjes <u>type</u> definition: for some nice functions f
for Ito integral

$$\int_a^b f(W(t), t)\, dW(t)$$

$$\equiv \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(W(t_{j-1}), t_{j-1})\left(W(t_j) - W(t_{j-1})\right)$$



Evaluate the function at <u>left endpoints</u>

~~Martingale~~

We will define the Itô Stochastic integral in a broader context which allows:

i) integrand does not have to be as nice.

ii) The integrand can depend on past and present values of $W(t)$, as well as other independent sources of randomness

So can consider integrals like

$$\int_a^b f\Big(\sup_{a \le s \le t-1} W(t),\ \underset{\substack{\uparrow \\ \text{other noise source}}}{Z(t)},\ t\Big)\, dW(t)$$

To prepare, we ~~define~~ assume we have a <u>filtration</u> $\{\mathcal{A}_t\}_{t \ge 0}$ such that

$W(t)$ and $f(t, \underset{\nwarrow \text{all randomness}}{\omega})$ are adapted to $\mathcal{A}_t$

and $W(t) - W(s)$ is independent of $\mathcal{A}_{s'}$ for $s' < s < t$.

We will define Itô stochastic integral in a way parallel to Lebesgue integration theory.

First consider simple functions of the form.

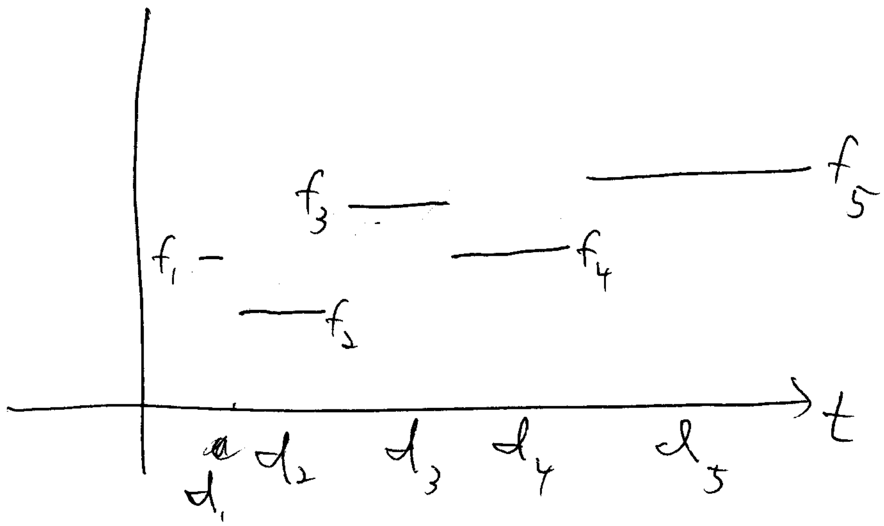
$$f(t,\omega) = \sum_{j=1}^{n} f_j(\omega) I\{t \in d_j\}$$

where d_j are disjoint intervals

$$d_j = [a_j, b_j]$$

and $\underline{\underline{f_j \in \mathcal{A}_{a_j}}}$ (Ito!)

$$\mathbb{E} f_j^2 < \infty$$



Call the space of these simple functions $\mathcal{S}$.

We will define a random functional

$J(f)$ which we will take as defn

of $$\int_{-\infty}^{\infty} f(\omega,t)\, dW(t)$$

For functions $f \in \mathcal{L}$, we clearly want:

$$J(f) = \sum_{j=1}^{n} f_j(\omega)(W(b_j) - W(a_j))$$

What properties does $J(f)$ have for $f \in \mathcal{L}$

1) Linearity: $J(\alpha f + \beta g) = \alpha J(f) + \beta J(g)$

for $f, g \in \mathcal{L}$, and $\alpha, \beta \in \mathbb{C}$

2) Mean of Itô stoch integral is always 0. for $f \in \mathcal{L}$

$$\mathbb{E}\, J(f) = \sum_{j=1}^{n} \mathbb{E}\left(f_j\,(W(b_j) - W(a_j))\right)$$

$$= \sum_{j=1}^{n} \mathbb{E}\left(\mathbb{E}\left(f_j\,(W(b_j) - W(a_j)) \,\middle|\, \mathcal{A}_{a_j}\right)\right)$$

$$= \sum_{j=1}^{n} \mathbb{E}\left(f_j\,\underbrace{\mathbb{E}\left(W(b_j) - W(a_j) \,\middle|\, \mathcal{A}_{a_j}\right)}_{= 0}\right)$$

$$= 0$$

3) Variance of Ito stochastic integral

$$\mathbb{E}(J(f)^2) = \sum_{j=1}^{n} \sum_{j'=1}^{n} \mathbb{E}(f_j f_{j'} (W(b_j) - W(a_j))(W(b_{j'}) - W(a_{j'}$$

$$= \sum_{j=1}^{n} \mathbb{E}(f_j^2 (W(b_j) - W(a_j))^2)$$

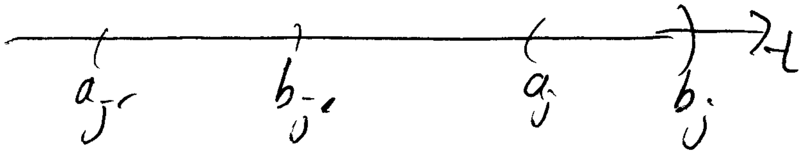
$$+ 2 \sum_{j > j'} \mathbb{E}(f_j f_{j'} (W(b_j) - W(a_j))(W(b_{j'}) - W(a_{j'})))$$

$$= \sum_{j=1}^{n} \mathbb{E}(\mathbb{E}(f_j^2 (W(b_j) - W(a_j))^2 \mid \mathcal{A}_{a_j})$$

$$+ 2 \sum_{j > j'} \mathbb{E}(\mathbb{E}(f_j f_{j'} (W(b_j) - W(a_j))(W(b_{j'}) - W(a_{j'})) \mid \mathcal{A}$$

(assuming intervals $d_1, d_2, \ldots, d_n$ are labelled in order from left to right)

$$= \sum_{j=1}^{n} \mathbb{E}(f_j^2 \, \mathbb{E}\underbrace{((W(b_j) - W(a_j))^2 \mid \mathcal{A}_{a_j})}_{= |b_j - a_j|})$$



$$+ 2 \sum_{j > j'} \mathbb{E}(f_j f_{j'} (W(b_{j'}) - W(a_{j'})) \underbrace{\mathbb{E}(W(b_j) - W(a_j) \mid \mathcal{A}_{a_j}}_{= 0}$$

$$= \sum_{j=1}^{n} \mathbb{E}(f_j^2) |b_j - a_j| = \int_{-\infty}^{\infty} (\mathbb{E} f^2) \, dt$$

Properties of ~~st~~ Itô stochastic integral

$$J(f) \text{ "="} \int_{-\infty}^{\infty} f(t,\omega)\, dW(t)$$

for simple functions $f \in \mathcal{L}$:

1) Linearity $J(\alpha f + \beta g) = \alpha J(f) + \beta J(g)$

 for $\alpha, \beta \in \mathbb{C}$, $f, g \in \mathcal{L}$

2) $\mathbb{E}\, J(f) = 0$

3) $\mathbb{E}\left(J(f)^2\right) = \int_{-\infty}^{\infty} \mathbb{E} f^2 \, dt$

4) $\mathbb{E}\left(J(f)\, J(g)\right) = \int_{-\infty}^{\infty} \mathbb{E}(fg)\, dt$

 - proof by polarization identity:

 $\mathbb{E}\left(J(f+g)^2 \right) - \mathbb{E}\left(J(f-g)^2 \right)$

How extend the defn of Ito integral to integrands which are not piecewise constant?

- idea is to approximate, in principle, the desired function by simple functions.

Let $\mathcal{L}^2$ denote random functions $f(t,\omega)$ such that

a) f measurable on $\mathbb{R} \times \Omega$

b) f is $\mathcal{a}_t$ adapted. (Ito!)

(non-anticipating functional)

c) $\int_{-\infty}^{\infty} \mathbb{E} f^2(t,\omega)\, dt < \infty$

$\mathcal{L}^2$ is a Hilbert space with norm

$$\|f\|_{\mathcal{L}^2} = \left(\int_{-\infty}^{\infty} \mathbb{E} f^2(t,\omega)\, dt \right)^{1/2}$$

Let $L^2(\Omega, \mathcal{a}, P)$ denote random variables with finite variance and zero mean.

This is a Hilbert space with norm

$$\|X\|_{L^2} = \left(\mathbb{E} X^2\right)^{1/2}$$

Then the properties 1) - 4) we derived for $J(f)$ show that

J is an <u>isometry</u> between

$$J : \mathcal{L} \longrightarrow L^2$$

One can show that $\mathcal{L}$ is a dense subspace of $\mathcal{L}^2$.

- Karatzas Platen Sec. 3.2

Therefore, $\mathcal{L}^2$ is completion of $\mathcal{L}$ under the $\mathcal{L}^2$ norm so the isometry $J : \mathcal{L} \to L^2$ can be extended in a unique way to an isometry $J : \mathcal{L}^2 \to L^2$.

- define $J(f)$ for $f \in \mathcal{L}^2$ as

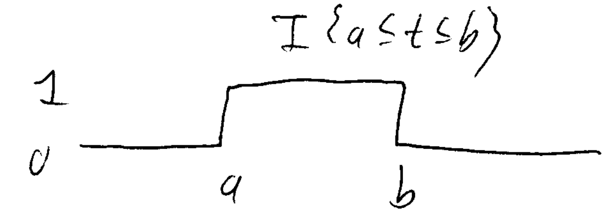
$$J(f) = \lim_{\substack{\|f_n - f\| \to 0 \\ f_n \in \mathcal{L}}} J(f_n)$$

All the properties 1) - 4) carry over to the Ito stochastic integral $J(f)$ defined on $\mathcal{L}^2$.

Further properties of Itô stochastic integral $\int_{-\infty}^{\infty} f(t,\omega)\,dW(t)$

1) Integrals over finite intervals

$$\int_a^b f(t,\omega)\,dW(t) = \int_{-\infty}^{\infty} f(t,\omega)\, I\{a \le t \le b\}\,dW(t)$$



2) $Z(t) = \int_a^t f(s,\omega)\,dW(s)$ is adapted to $\mathcal{A}_t$.
(by construction)

3) $Z(t)$ can be assumed continuous function of t
(maximal martingale ineqs)

4) $Z(t)$ is a strong Markov process w.r.t. $\mathcal{A}_t$.

5) $Z(t) = \int_a^t f(s,\omega)\,dW(s)$ is a martingale.

Proof: Consider $Z(t) - Z(t') = \int_{t'}^{t} f(s,\omega)\, dW(s)$

$$\mathbb{E}\left(Z(t) - Z(t') \mid \mathcal{a}_{t'}\right)$$

for simple fns $= \mathbb{E}\left(\sum_{j=1}^{n} f_j(\omega)\left(W(b_j) - W(a_j)\right) \mid \mathcal{a}_{t'}\right)$

where $t' = a_0 < b_0 \leq a_1 < b_1 \cdots \leq a_n < b_n = t$

$$= \sum_{j=1}^{n} \mathbb{E}\left(\mathbb{E}\left(f_j(\omega)\left(W(b_j) - W(a_j)\right) \mid \mathcal{a}_{a_j}\right) \mid \mathcal{a}_{t'}\right)$$

$$= \sum_{j=1}^{n} \mathbb{E}\left(f_j \underbrace{\mathbb{E}\left(W(b_j) - W(a_j) \mid \mathcal{a}_{a_j}\right)}_{=0} \mid \mathcal{a}_{t'}\right)$$

$$= 0$$

$$\mathbb{E}\left(Z(t) \mid \mathcal{a}_{t'}\right) = Z(t') \text{ for } t > t'$$

~~Check $\mathbb{E}$~~ True also for $f \in \mathcal{L}^2$ by taking limits.

$$\mathbb{E}\,|Z(t)| \leq \left(\mathbb{E}\, Z^2(t)\right)^{1/2} < \infty$$

Hölder ineq

Now with a stochastic integration theory, can define an Ito SDE:

$$dX(t,\omega) = e(t,\omega)\,dt + f(t,\omega)\,dW(t)$$

with $\sqrt{|e|}, f \in \mathcal{L}^2$

The solution to such an SDE is defined to be a soln to Ito stochastic integral eqn

$$X(t,\omega) - X(s,\omega) = \int_s^t e(u,\omega)\,du + \int_s^t f(u,\omega)\,dW(u)$$