04/19/04

For the opposite extreme of pure diffusion:

$v \equiv 0$ (no drift)

$K = \mathbb{1}\frac{1}{2}$ (pure constant diffusion)

identity matrix

Backward equation:

$$-\frac{\partial u}{\partial s} = \frac{1}{2}\Delta u \qquad \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots \frac{\partial^2}{\partial x_d^2}$$

$$u(\vec{x}, s=t) = f(\vec{x})$$

is one way to solve for

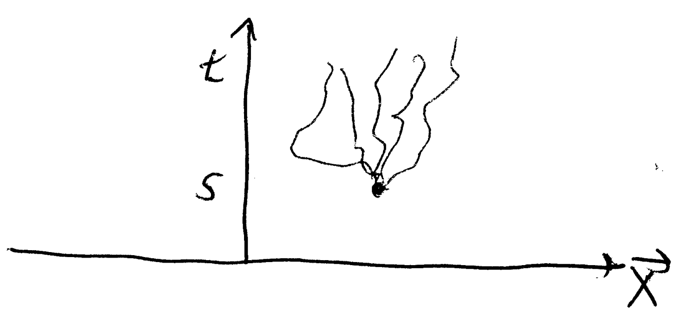
$$u(\vec{x}, s) = \mathbb{E}\left(f(\vec{x} + \vec{W}(t-s))\right)$$

$$= \mathbb{E}\left(f(\vec{X}(t)) \,\middle|\, \vec{X}(s) = \vec{x}\right)$$

where $\vec{X}(t) = \vec{x} + \vec{W}(t-s)$

$$\vec{W}(t) = \begin{pmatrix} W_1(t) \\ W_2(t) \\ \vdots \\ W_d(t) \end{pmatrix}$$ and each $W_j(t)$ is an independent Wiener process

Like a random MOC



For a general diffusion process w/ variable drift and diffusion, one might guess the appropriate diff eqn to write down is:

$$\frac{d\vec{X}}{dt} = \underbrace{\vec{v}(\vec{X}(t), t)}_{\substack{\text{drift} \\ \text{mean trend}}} + \underbrace{\sigma(\vec{X}(t), t)}_{\text{diffusivity / volatility}} \frac{d\vec{W}}{dt}$$

But $\frac{d\vec{W}}{dt}$ doesn't exist!

Write instead

$$\boxed{d\vec{X} = \vec{v}(\vec{X}(t), t)\,dt + \sigma(\vec{X}(t), t)\,d\vec{W}(t)}$$

Stochastic Differential Equation

There are hidden subtlety in the $d\vec{W}$ term.

Examples of SDE's:

1) Particle moving through a low Reynolds # fluid w/ thermal flucs

$$m\, d\vec{v} = \underbrace{-\gamma \vec{v}\, dt}_{\text{friction}} + \underbrace{\sigma\, d\vec{W}}_{\text{thermal}}$$

$\vec{v}$ = velocity

m = mass

γ = friction constant

σ = noise parameter

Physicist notation (sometimes)

$$m \frac{d\vec{v}}{dt} = -\gamma \vec{v} + \vec{f}_T(t)$$

thermal force

$$\langle \vec{f}_T(t) \otimes \vec{f}_T(t') \rangle = \sigma^2 \mathbb{1}\, \delta(t-t')$$

$$\langle \vec{f}_T(t) \rangle = 0$$

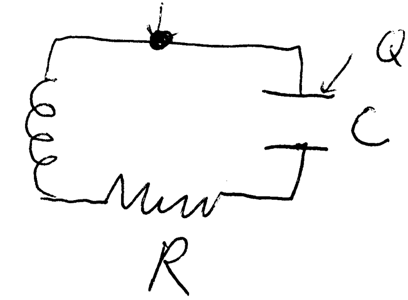
"white noise"

2) Particle with added external potential $U(\vec{x})$

$$d\vec{X} = \vec{v}\, dt$$

$$m\, d\vec{v} = \underbrace{-\gamma \vec{v}\, dt}_{\text{friction}} - \underbrace{\vec{\nabla} U(\vec{x})\, dt}_{\text{force from potential}} + \underbrace{\sigma\, d\vec{W}(t)}_{\text{thermal}}$$

3) LRC circuit

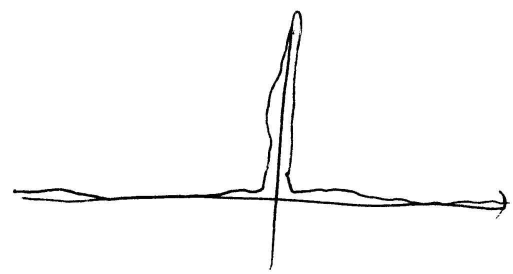


$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + CQ = V_{ext}(t) + \tilde{V}(t)$$

Q = charge on capacitor

V_{ext} = applied voltage

$\tilde{V}$ = thermal flucs



"White noise" fluctuations: $\langle \tilde{V}(t)\tilde{V}(t') \rangle = \sigma^2 \delta(t-t')$

SDE formalism:

$$dQ = I\,dt$$

$$L\,dI + RI\,dt + CQ\,dt = \sigma\,dW(t) + V_{ext}(t)\,dt$$

4) Turbulent transport

$$d\vec{X} = \vec{u}(\vec{X}(t), t)\,dt + \underbrace{\sqrt{2K}\,d\vec{W}(t)}_{\text{molecular/unresolved velocity}}$$

$\vec{X}(t)$ = position of a tracer

$\vec{u}(\vec{x}, t)$ = flow of fluid

K is like eddy diffusivity

$K = K(\vec{x}, t)$

5) Finance

Black-Scholes model (geometric Brownian motion)

$X(t)$ = asset price

$$dX = \mu X\,dt + \underline{\sigma X\,dW(t)}$$

mean rate of growth = μ

volatility is proportional to current price

6) Stochastic filtering: (<u>Oksendal</u>)

Model for how to interpret noisy measurements of an unpredictable system.

System Model: $$dX = b(t, X(t))\,dt + \overset{\substack{\text{unpredictability} \\ \downarrow}}{\sigma(t, X(t))}\,dW_1(t)$$

Observation Model: $$dZ = c(t, X(t))\,dt + \underbrace{\gamma(t, X(t))\,dW_2(t)}_{\text{error in measurements}}$$

$dW_1(t)$ and $dW_2(t)$ are independent

General principle:

The stochastic diff eq (SDE)

$$d\vec{X} = \vec{v}(\vec{X}(t),t)\,dt + \sigma(\vec{X}(t),t)\,d\vec{W}(t)$$

with the "Itô" interpretation

generates a Markov process which is a diffusion process with drift $\vec{v}(\vec{x},t)$ and diffusivity $K(\vec{x},t) = \frac{1}{2}\sigma\sigma^{+}$

and infinitesimal generator

$$A = \vec{v}(\vec{x},t)\cdot\vec{\nabla} + K(\vec{x},t):\vec{\nabla}\vec{\nabla}$$

Many questions involving $\vec{X}(t)$ can be solved in terms of PDE's involving A.

For example: i) Prob trans. densities $p(s,\vec{x};t,\vec{y})$

Kol back: $-\frac{\partial}{\partial s}p = A_{\vec{x},s}\,p$ Kol forw: $\frac{\partial p}{\partial t} = A^{*}_{\vec{y},t}\,p$

ii) Suppose that I accumulate a cost/reward at a rate $f(\vec{x})$ where $\vec{x} \in S$.

$$\vec{w}(\vec{x}, t) = \mathbb{E}\left[\int_0^t f(\vec{X}(s))\,ds \;\middle|\; \vec{X}(0) = \vec{x}\right]$$

Accumulated cost/reward up to time t, given initial state $\vec{x}$.

This is the solution to parabolic PDE:

$$\frac{\partial \vec{w}}{\partial t} = A\vec{w} + \vec{f}$$

$$\vec{w}(t=0) = \vec{0}$$

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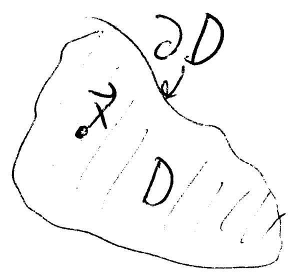
iii) Suppose the state space has a bounded domain D with bdry ∂D



Let τ be first time that $\vec{X}(t)$ hits ∂D.

Consider

$$\vec{v}(\vec{x}) = \mathbb{E}\left[\int_0^\tau \vec{f}(\vec{X}(s))\,ds + \vec{g}(\vec{X}(\tau)) \;\middle|\; \vec{X}(t=0) = \vec{x}\right]$$

This satisfies elliptic PDE

$$A\vec{v} + \vec{f} = 0 \quad \text{for } \vec{x} \in D$$

$$\vec{v} = \vec{g} \quad \text{for } \vec{x} \in \partial D$$

Monte Carlo Methods M. Mascagni K. Sabelfeld

What's the difficulty with making sense of SDE?

$$d\vec{X} = \vec{v}(\vec{X}(t), t)\,dt + \sigma(\vec{X}(t), t)\,d\vec{W}(t)?$$

If I write it in integral form:

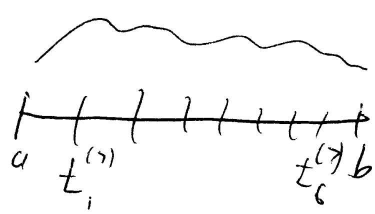
$$\vec{X}(t) - \vec{X}(0) = \int_0^t \vec{v}(\vec{X}(s), s)\,ds + \underbrace{\int_0^t \sigma(\vec{X}(s), s)\,d\vec{W}(s)}_{\text{How define?}}$$

Lebesgue integration doesn't work because ~~dW~~ $\vec{W}(t)$ doesn't have <u>bounded variation</u>.

What about Riemann-Stieltjes interpretation?

Consider an integral like

$$\int_a^b f(W(t), t)\,dW(t)$$



Define a partition $a = t_0^{(n)} < t_1^{(n)} < \dots < t_n^{(n)} = b$

The Riemann-Stieltjes sum:

$$\sum_{j=1}^{n} f(W(\tau_j^{(n)}), \tau_j^{(n)})\,(W(t_j^{(n)}) - W(t_{j-1}^{(n)}))$$

where $t_{j-1}^{(n)} \le \tau_j^{(n)} \le t_j^{(n)}$ does not give unique answer!