04/15/04 More general diffusion processes
Kloeden + Platen Sec. 1.7

Allow the continuous-time, continuous-state stochastic process to have a drift term and diffusion term, both of which can depend on time and on state of the process
- only consider <u>continuous</u> SP

Intuitively:

Drift $$v(x,s) = \lim_{t \downarrow s} \frac{1}{t-s} \mathbb{E}\left(X(t) - X(s) \middle| X(s) = x\right)$$

Diffusivity: $$\kappa(x,s) = \lim_{t \downarrow s} \mathbb{E}\left(\frac{(X(t)-X(s))^2}{2(t-s)} \middle| X(s) = x\right)$$

For Wiener process $v(x,s) \equiv 0$, $\kappa(x,s) = \frac{1}{2}$

Technically, a diffusion process $X(t)$ taken values in $S = \mathbb{R}$ such that its ~~prob. trans~~ transition probability density

$$P(s, x; t, dy)$$

Satisfies:

a) $$\lim_{t \downarrow s} \frac{1}{t-s} \int_{|y-x|>\varepsilon} P(s,x;t,dy) = 0 \quad \text{for any } \varepsilon > 0$$

(excludes discontinuous jumps)

b) $$\lim_{t \downarrow s} \frac{1}{t-s} \int_{|y-x|<\varepsilon} (y-x)\, P(s,x;t,dy) = v(x,s)$$

c) $$\lim_{t \downarrow s} \frac{1}{2(t-s)} \int_{|y-x|<\varepsilon} (y-x)^2\, P(s,x;t,dy) = K(x,s)$$

Recall: $P(x, s; t, B) = \text{Prob}(X(t) \in B \mid X(s) = x)$

Diffusion processes can be shown to have a transition probability <u>density</u> provided $K(x,s) \geq m > 0$

$$P(x; s; t, B) = \int_B p(x,s;t,y)\,dy$$

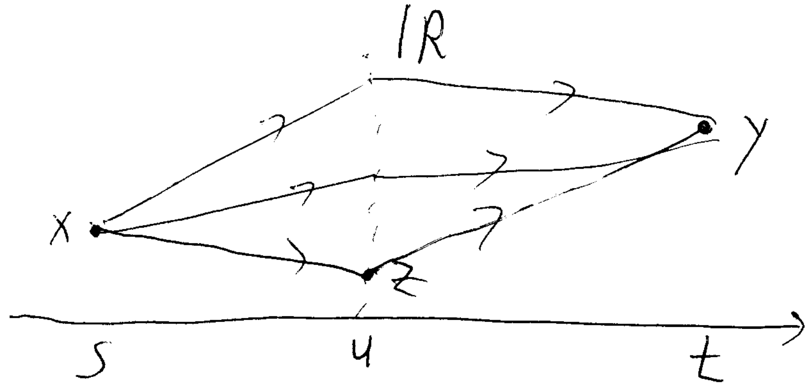
$$P(x,s;t,dy) = p(x,s;t,y)\,dy$$

Kolmogorov backward equation for a diffusion process.

- one way to solve for prob. trans. density

Chapman-Kolmogorov eqn:

$$p(s,x;t,y) = \int dz \; p(s,x;u,z)\, p(u,z;t,y)$$




To derive backward eqn:

$$p(s-\Delta s,x;t,y) = \int_{\mathbb{R}} dz \; p(s-\Delta s,x;s,z)\, p(s,z;t,y)$$

Assuming smoothness

$$p(s,z;t,y) = p(s,x;t,y) + (z-x)\frac{\partial}{\partial x}p(s,x;t,y) + \frac{1}{2}(z-x)^2\frac{\partial^2}{\partial x^2}p(s,x;t,y) + O(|z-x|^3)$$

Sub into integral

$$p(s-\Delta s, x; t, y) = \left( \int_{\mathbb{R}} dz \; \underbrace{p(s-\Delta s, x; s, z)}_{1} \right) p(s, x; t, y)$$

$$+ \left( \int_{\mathbb{R}} dz \, (z-x) \, p(s-\Delta s, x; s, z) \right) \frac{\partial}{\partial x} p(s, x; t, y)$$

$$+ \frac{1}{2} \left( \int_{\mathbb{R}} dz \, (z-x)^2 \, p(s-\Delta s, x; s, z) \right) \frac{\partial^2}{\partial x^2} p(s, x; t, y)$$

$$+ \int_{\mathbb{R}} dz \, O(z-x)^3 p(s-\Delta s, x; s, z) \; \frac{\partial^3}{\partial x^3} p(s, \theta; t, y)$$

with $x \leq \theta \leq z$

As $\Delta s \to 0$:

$$p(s-\Delta s, x; t, y) = p(s, x; t, y) + v(x, s) \Delta s \frac{\partial}{\partial x} p(s, x; t, y)$$

$$+ K(x, s) \Delta s \frac{\partial^2}{\partial x^2} p(s, x; t, y) + \underbrace{o(\Delta s)}_{\ll \Delta s}$$

$$\frac{p(s-\Delta s, x; t, y) - p(s, x; t, y)}{\Delta s} = v(x, s) \frac{\partial}{\partial x} p(s, x; t, y)$$

$$+ K(x, s) \frac{\partial^2}{\partial x^2} p(s, x; t, y)$$

$$+ o(1)$$

Taking $\Delta s \to 0$ limit:

$$\boxed{-\frac{\partial p(s,x;t,y)}{\partial s} = v(x,s)\frac{\partial}{\partial x}p(s,x;t,y) + K(x,s)\frac{\partial^2 p(s,x;t,y)}{\partial x^2}}$$

Kolmogorov backward equation for diffusion process

(Initial data: ~~p(s,x;t,y)~~ $p(s,x;t=s,y) = \delta(x-y)$)

Infinitesimal generator

$$A = v(x,s)\frac{\partial}{\partial x} + K(x,s)\frac{\partial^2}{\partial x^2}$$

(discretization gives random walk w/ bias)

Multi-dims:

$$A = \vec{v}(\vec{x},s)\cdot\vec{\nabla} + \mathcal{K}(\vec{x},s) : \vec{\nabla}\vec{\nabla}$$

Recall that the Kolmogorov backward eqn is also the equation which

$$u(s,x) = \mathbb{E}\left(f(X(t)) \mid X(s) = x\right)$$

satisfies, but with "initial" data

$$u(t,x) = f(x)$$

Forward Kolmogorov equation for diffusion processes

If try parallel argument by using Chapman-Kolmogorov eqn with times $s, t, t+\Delta t$, it doesn't work.

Instead consider

~~$p(s, x; t, y)$~~

$$\frac{p(s,x;t+\Delta t,y)-p(s,x;t,y)}{\Delta t} =$$

$$\int_{\mathbb{R}} dz \; p(s,x;t,z)\left(\frac{p(t,z;t+\Delta t,y)-\delta(z-y)}{\Delta t}\right)$$

$$\uparrow \text{C-K}$$

$$\overbrace{\underbrace{\frac{p(t,z;t+\Delta t,y)-p(t+\Delta t,z;t+\Delta t,y)}{\Delta t}}}$$

As $\Delta t \to 0$ $\qquad \downarrow$

$$-\frac{\partial p}{\partial s}(s, z; t+\Delta t, y)\Big|_{s=t+\Delta t}$$

$$\shortparallel$$

backward Kolmogorov $\longrightarrow$ $\left(v(z, t+\Delta t)\frac{\partial}{\partial z} + K(z,t+\Delta t)\frac{\partial^2}{\partial z^2}\right) p(t+\Delta t, z; t+\Delta t, y)$

So, as $\Delta t \to 0$

$$\frac{\partial}{\partial t} p(s,x;t,y) = \int_{\mathbb{R}} dz \; p(s,x;t,z) \times$$

$$\left[ v(z,t) \frac{\partial}{\partial z} + K(z,t) \frac{\partial^2}{\partial z^2} \right]$$

$$p(t;z;t,y)$$

$$p(t,z;t,y) = \delta(z-y)$$

Integrate by parts:

$$\frac{\partial}{\partial t} p(s,x;t,y) = \int_{\mathbb{R}} dz \left[ -\frac{\partial}{\partial z}\big(v(z,t)\,p(s,x;t,z)\big) + \frac{\partial^2}{\partial z^2}\Big(K(z,t)\,p(s,x;t,z)\Big) \right] \delta(z-y)$$

$$\boxed{\frac{\partial}{\partial t} p(s,x;t,y) = -\frac{\partial}{\partial y}\Big(v(y,t)\,p(s,x;t,y)\Big) + \frac{\partial}{\partial y^2}\Big(K(y,t)\,p(s,x;t,y)\Big)}$$

Kolmogorov forward equation

(Initial data: $p(s,x;t=s,y) = \delta(x-y)$ )

Notice Kolmogorov forward equation has the form $\frac{\partial}{\partial t} p = A^* p$

adjoint

Backward eqn: $-\frac{\partial}{\partial s} p = A p$

Recall also that if $\phi(y,t)$ is the probability density for the process

$\left( \text{Prob}( X(t) \in B) = \int_B dy \; \phi(y,t) \right)$

then $\phi$ satisfies the Kolmogorov forward eqn.

Connection: $\mathbb{E}\, f(X(t) \mid X(s) = x)$

$$= \int dy \; \phi(y,t) f(y)$$

with $\phi(y,s) = \delta(x-y)$

The various formulas involving absorption probabilities, integrated payoffs, stationary dists from continuous-time MC all carry over with just changing infinitesimal generator. $\boxed{\text{Reichl Ch. 6}}$

But what about <u>path</u> properties of a stochastic process? How simulate?

- think about diffusion processes in terms of <u>stochastic differential eqn</u>

Simple cases:

I) No diffusion $K \equiv 0$

Kolmogorov backward eqn for $u(s,\vec{x}) = \mathbb{E}\left[f(\vec{X}(t)) \,\middle|\, \vec{X}(s)=\vec{x}\right]$

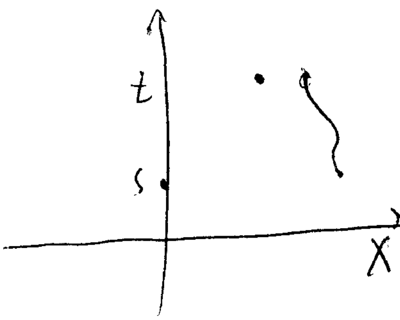
$$-\frac{\partial u}{\partial s} = \vec{v}(\vec{x},s)\cdot\vec{\nabla} u$$

$$u(s=t,\vec{x}) = f(\vec{x})$$

Solve: Method of characteristics:

Associated trajectory $\vec{X}(t)$

which satisfies: $\dfrac{d\vec{X}(t)}{dt} = \vec{v}(\vec{X}(t),t)$




Solve this eqn ~~with the get~~ with $\vec{X}(s)=\vec{x}$

$$\vec{X}(t) = \vec{x} + \int_s^t \vec{v}(\vec{X}(u),u)\,du$$

Then $u(s,\vec{x}) = f\left(\vec{x} + \int_s^t \vec{v}(\vec{X}(u),u)\,du\right)$

solves the Kolmogorov backward eqn.