How do we calculate statistics of Z?

Clever recursive argument.

Relate epidemic statistics for a given population to those of a population with a smaller # of initial susceptibles.

Let $Z^{n'}$, $\Phi^{n'}$ be the total epidemic size and infection pressure with n' initial susceptibles

but always normalize infection rate by $\frac{1}{n+m}$

$\uparrow$ not n'

$$\Phi^{n'} = \frac{\lambda}{n+m} \int_0^t I^{n'}(s)\, ds$$

$$S^{n'}(0) = n'$$

$$I^{n'}(0) = m$$

04/08/04 : HW 3 due 04/14 Wed at 5 PM

HW 4 (last) : look for over wknd

- due ~~00~~ Wednesday April 28

(maybe a little later)

≥ 360	A
280-359	B
200-279	C

No PK OH on 04/13

"Take Your Students to Lunch"

- contact me kramep@rpi.edu

Tedious calculation for PDF for Z:

$$p_k^{n'} = \text{Prob}(Z^{n'} = k)$$

$\tilde{p}_k^{n'}$ = Prob(The individuals infected, with n' initial susceptibles are precisely those labeled $\{1, 2, \ldots, k\}$)

Recursion: Choose $k \le \ell \le n'$

$$\tilde{p}_k^{n'} = \tilde{p}_k^{\ell}\, \mathbb{E}\left(\exp(-\Phi^{\ell}(n'-\ell)) \mid Z^{\ell} = k\right)$$

Why? LHS = Prob(Individuals $1 \ldots k$ out of n' get infected but inds. $k+1 \ldots n'$ don't)

$= \underbrace{\text{Prob(Individuals } 1 \ldots k \text{ out of } \ell \text{ people get infected)}}_{\tilde{p}_k^{\ell}}$

Prob(Individuals $\ell+1, \ldots, n'$ not infected | individuals $1 \ldots k$ out of $1 \ldots \ell$ are the ones infected)

Each individual $\ell+1, \ldots, n'$ has probability $e^{-\Phi^{\ell}}$ not to get infected by the k out of ℓ, Let us study the conditional expectation

$$\Phi^\ell = \frac{\lambda}{n+m} \sum_{j=-(m-1)}^{Z^\ell} \tau_j \quad \text{: random sum}$$

$$Z^\ell = \min\left\{ i : Q_{(i+1)} > \frac{\lambda}{n+m} \sum_{j=-(m-1)}^{i} \tau_j \right\}$$

Can't evaluate the cond exp by just subbing in $Z^\ell = k$ because Z^ℓ depends on the τ_j, and knowing that Z^ℓ attains some value biases the PDF's for τ_j.

But: Z^ℓ only depends on the τ_j with $j \leq Z^\ell$

So it's like a Markov time w.r.t. the rvs $\{\tau_j\}_{j=-(m-1)}^{n}$

Markov times are good when systems have "strong Markov property" or when one can construct martingales

Martingales (Karlin + Taylor Sec. 6.1–6.5)

Remember a filtration is an increasing family of σ-algebras $\mathcal{A}_t$

$\mathcal{A}_t$ is like the collection of events which are determined up to and including time t.

Martingale w.r.t. filtration $\mathcal{A}_t$
$Y(t)$

is a stoch. process such that

$$\mathbb{E}\left(Y(t) \mid \mathcal{A}_s \right) = Y(s) \quad \text{for } s \le t$$

and

$$\mathbb{E}\left(|Y(t)| \right) < \infty$$

A more concrete way to write martingale property is that if A is an event $A \in \mathcal{A}_s$ (only involves info about $Y(t)$ up to and including $t = s$) then

$$\mathbb{E}\left(Y(s') \mid A \text{ and } Y(s) = i \right) = i$$

for $s \le s'$

Examples of martingales:

Let $X_1, \dots, X_n$ be ~~iid~~ independent rv's with $\langle |X_j| \rangle < \infty$

Then $$Y_n = \sum_{j=1}^{n} \left(X_j - \langle X_j \rangle \right)$$

is a martingale w.r.t. the filtration generated by $\{X_n\}$

What about ~~variance~~ square of the sum?

Let $V_n = \left(\sum_{j=1}^{n} X_j\right)^2 - \sum_{j=1}^{n} \sigma_j^2$

with $\{X_j\}$ being ~~iid~~ independent

$\langle X_j^2 \rangle = \sigma_j^2$; $\langle X_j \rangle = 0$

Then can check

$$\mathbb{E}(V_{n+1} \mid \mathcal{a}_n) = V_n$$

so V_n is a martingale w.r.t. the filtration $\mathcal{a}_n$ generated by $\{X_n\}$,

$$\mathbb{E}(V_{n+1} \mid \mathcal{a}_n) = \mathbb{E}\left(\left(\sum_{j=1}^{n+1} X_j\right)^2 - \sum_{j=1}^{n+1} \sigma_j^2 \,\middle|\, \mathcal{a}_n\right)$$

$$= \mathbb{E}\left[\left(\sum_{j=1}^{n} X_j\right)^2 - \sum_{j=1}^{n} \sigma_j^2 + X_{n+1}^2 + 2X_{n+1}\sum_{j=1}^{n} X_j - \sigma_{n+1}^2 \,\middle|\, \mathcal{a}_n\right]$$

$$= \mathbb{E}\left[V_n + X_{n+1}^2 - \sigma_{n+1}^2 + 2X_{n+1}\sum_{j=1}^{n} X_j \,\middle|\, \mathcal{a}_n\right]$$

$$= V_n + \cancel{\sigma_{n+1}^2} - \cancel{\sigma_{n+1}^2} + 2\sum_{j=1}^{n} X_j \, \underbrace{\cancel{\mathbb{E}(X_{n+1} \mid \mathcal{a}_n)}}_{0}$$

Branching process: $X_{n+1} = \sum_{j=1}^{X_n} Z_{n,j}$

where $Z_{n,j}$ are iid rvs, $\langle Z_{n,j} \rangle = m$

Multiplicative normalization:

$$Y_n = \frac{X_n}{m^n}$$

$$\mathbb{E}[Y_{n+1} | \mathcal{a}_n] = \mathbb{E}\left[\frac{\sum_{j=1}^{X_n} Z_{n,j}}{m^{n+1}} \middle| \mathcal{a}_n \right]$$

$$= \sum_{j=1}^{X_n} \frac{\mathbb{E}[Z_{n,j} | \mathcal{a}_n]}{m^{n+1}}$$

$$= \sum_{j=1}^{X_n} \frac{m}{m^{n+1}}$$

$$= \frac{X_n}{m^n} = Y_n$$

So Y_n is a martingale w.r.t. filtration $\mathcal{a}_n$ generated by the $\{X_n\}$.

What's the use of martingales?

Mathematical: control fluctuations and prove smoothness

control extreme values:

For example

$$\text{Prob}\{\max_{0 \le k \le n} X_k > a\} \le \frac{\mathbb{E}(|X_n|)}{a}$$

for martingales

At long times martingales converge to random variables

Computational purposes: relation to Markov times

Optional Stopping Theorem:

If $Y(t)$ is martingale and T is Markov time with $Pr(T<\infty) = 1$,

$\mathbb{E}(\sup_{t \ge 0} |Y(t)|) < \infty$ then

$$\mathbb{E}(Y(T)) = \mathbb{E}(Y(0))$$

One important application is to calculating random sums where the number of terms in the sum is a Markov time.

<u>Wald identity:</u> (K+T Sec. 6.4)

Let $\{X_n\}$ be iid, nonnegative rv's, $S_0 = 0$, $S_n = \sum_{j=1}^{n} X_j$ and T is a Markov time w.r.t. the filtration $\mathcal{U}_n$ generated by $\{X_n\}$ and which satisfies $T \leq N$ for some $N < \infty$.

Let $\mathcal{L}_X(\theta) = \langle e^{-\theta X} \rangle$ be moment gen fn.

Then
$$\mathbb{E}\left(\frac{\exp(-\theta S_T)}{\mathcal{L}_X(\theta)^T} \right) = 1$$

Proof: $Y_n = \dfrac{e^{-\theta S_n}}{\mathcal{L}_X(\theta)^n}$ for $0 \leq n \leq N$, $Y_n = Y_N$ for $n \geq N$

is a bounded martingale: use Opt. Stop Thm.

Applying Wald identity to the recursion formula w/ the cond. exp., we'd get

$$\mathbb{E}\left(\frac{\exp(-\Phi^{\ell}(n'-\ell))}{\mathcal{L}_{I}\left(\frac{\lambda(n'-\ell)}{n+m}\right)^{Z_\ell+m}}\right) = 1$$

$$\parallel$$

$$\sum_{k=0}^{\ell} \mathbb{E}\left(\frac{\exp(-\Phi^{\ell}(n'-\ell))}{\left(\mathcal{L}_{I}\left(\frac{\lambda(n'-\ell)}{n+m}\right)\right)^{Z_\ell+m}} \,\middle|\, Z^{\ell}=k\right)$$

$$\times \operatorname{Prob}(Z^{\ell}=k) = 1$$

$$\sum_{k=0}^{\ell} \frac{\mathbb{E}(\exp(-\Phi^{\ell}(n'-\ell)) \mid Z^{\ell}=k)\,\underbrace{\operatorname{Prob}(Z^{\ell}=k)}_{=p^{\ell}_{k}}}{\mathcal{L}_{I}\left(\frac{\lambda(n'-\ell)}{n+m}\right)^{k+m}} = 1$$

$$\sum_{k=0}^{\ell} \frac{\tilde{p}^{n'}_{k}}{\tilde{p}^{\ell}_{k}} \frac{p^{\ell}_{k}}{\left(\mathcal{L}_{I}\left(\frac{\lambda(n'-\ell)}{n+m}\right)\right)^{k+m}} = 1$$

$$p_k^{n'} = \binom{n'}{k} \hat{p}_k^{n'}$$

After some factorial reduction

$$\sum_{k=0}^{\ell} \frac{\binom{n'-k}{\ell-k}}{\binom{n'}{\ell}} \, p_k^{n'} \varrho_I \left(\frac{\lambda(n'-\ell)}{n+m} \right)^{-k-m} = 1$$

for $0 \leq \ell \leq n'$

Triangular system for p_k^n ... recursively solvable

p_0^0