04/05/04 : HW3 due date extended
04/14 at 5 PM
- PK: no OH on 04/13

How long (in terms of time variable $t$) does a new request have to wait before it is met by the server.

Let this random quantity be $T$.

$$T = \sum_{j=0}^{N} T_j \quad \text{where } N = \# \text{ demands in the queue upon arrival of new request}$$

$T_j$ = random time to fulfill request of $j$th "customer"

$j=0$: new demand

$\{T_j\}$ are i.i.d so $T$ is a random sum of iid r.v.'s

Use generating function for $T$:

$$\mathcal{L}_T(\theta) \equiv \mathbb{E}\, e^{-\theta T} = \int_0^\infty e^{-\theta t} p_T(t)\,dt$$

$p_T$: pdf for $T$

(Laplace transform for $p_T$; works basically same way as other versions of gen fn)

$$\mathcal{L}_T(\theta) = \mathbb{E}\, e^{-\theta \sum_{j=0}^{N} T_j} = \langle e^{-\theta \sum_{j=0}^{N} T_j} \rangle$$

$$= \langle \langle e^{-\theta \sum_{j=0}^{N} T_j} \rangle_T \rangle_N$$

$$= \langle (\mathcal{L}_{T_1}(\theta))^{N+1} \rangle$$

where $\mathcal{L}_{T_1}(\theta) = \langle e^{-\theta T_1} \rangle$

$$= G_N(\mathcal{L}_{T_1}(\theta))\, \mathcal{L}_{T_1}(\theta)$$

where $G_N(s) = \langle s^N \rangle$

$$p_{T_1}(t) = \mu e^{-\mu t} \quad \text{for } t \geq 0$$

$$\mathcal{L}_{T_1}(\theta) = \int_0^\infty \mu e^{-\mu t} e^{-\theta t}\, dt$$

$$= \frac{\mu}{\mu + \theta}$$

Assume system has settled down to stat dist. (with $\lambda < \mu$)

$$G_N(s) = \langle s^N \rangle = \sum_{j=0}^{\infty} \pi_j s^j = \sum_{j=0}^{\infty} \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^j s^j$$

$$= \frac{1 - \frac{\lambda}{\mu}}{1 - \frac{\lambda s}{\mu}}$$

$$\mathcal{L}_T(\theta) = \frac{\left(1-\frac{\lambda}{\mu}\right)\frac{\mu}{\mu+\theta}}{1-\frac{\lambda}{\mu}\left(\frac{\mu}{\mu+\theta}\right)} = \frac{\left(1-\frac{\lambda}{\mu}\right)\frac{\mu}{\mu+\theta}}{1-\frac{\lambda}{\mu+\theta}}$$

$$= \frac{\mu-\lambda}{\mu+\theta-\lambda} = \frac{\mu-\lambda}{\mu-\lambda+\theta}$$

which is Laplace xform of exp dist.

$T$ has pdf

$$p_T(t) = (\mu-\lambda)\, e^{-(\mu-\lambda)t}$$

exp. dist. with mean $\frac{1}{\mu-\lambda}$.

# Stochastic Models for Epidemics

Standard deterministic model for contagious person-to-person disease (Hethcote, SIAM Review 2000)

$S$ = susceptible

$I$ = infected

$R$ = recovered (immune)

Kermack-McKendrick model:

$$\frac{dS}{dt} = -\frac{\lambda S I}{N}$$

$\lambda$ = per capita contact rate

$N = S + I + R$

$$\frac{dI}{dt} = \frac{\lambda S I}{N} - \gamma I$$

$\gamma$ = rate of recovery

$$\frac{dR}{dt} = \gamma I$$

Often in a highly contagious disease, there can be a wide variety of outcomes from given initial data.

- slight variations in initial progress disease can amplify to large changes

Simple stochastic model:

Reed-Frost model:

$\boxed{\text{Andersson-Britton Sec. 1.2}}$

Discrete-time MC model for $(S, I)$

$$R = N - S - I$$

At each epoch, each infected person has probability $p$ of infecting each susceptible person,

$\frac{1}{\gamma}$ = mean time of infection

$$p \approx 1 - e^{-\lambda/\gamma N}$$

(Prob(no infection from the I during time t

$$\approx e^{-\lambda t/N}$$

At each epoch, all the infected people recover.

(Each epoch corresponds to a "generation" of the disease spread.)

$q = 1-p$

$$\text{Prob}(S_{n+1} = s_{n+1}, I_{n+1} = i_{n+1} \mid S_n = s_n, I_n = i_n)$$

$$= \binom{s_n}{i_{n+1}} (1 - q^{i_n})^{i_{n+1}} \left(q^{i_n}\right)^{s_n - i_{n+1}}$$

Prob a given S does not get sick

if $s_{n+1} + i_{n+1} = s_n$

$= 0$ otherwise

This is used a bit but continuous-time stochastic models for epidemics also of interest for more flexibility in handling extra complexity

- variability in recovery time
- when introduce other effects like vaccination, death, immigration, etc., these superpose simply in instantaneous sense but not simply over finite time intervals.

Continuous-time stochastic SIR model $\boxed{\text{Andersson-Britton Ch. 2}}$

Each infected person communicates disease to each susceptible person according to a Poisson process w/ rate $\frac{\lambda}{n+m}$

$\lambda$ = per capita contact rate

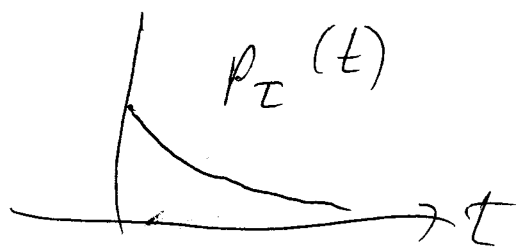
$n$ = # susceptibles initially $\qquad N = n+m$

$m$ = # infected initially

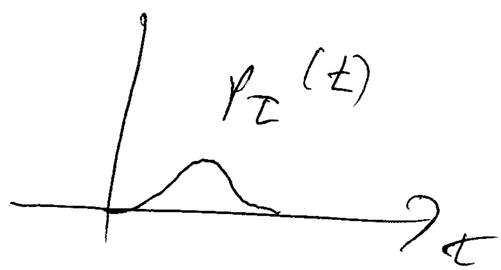
All interactions between people occur independently.

Each infected person remains infective for a random time $I$, which is i.i.d for different infections.

Is this a continuous-time MC?
- only if $I$ has exp dist.




- other distributions of interest to better match data




- $I$ deterministic $\Rightarrow$ Reed-Frost model

For Markovian case ( ~~I ~ exp~~ $p_I(t) = \gamma e^{-\gamma t}$ )

Transition rules: $(s,i) \to (s-1, i+1)$ $\quad \frac{\lambda s i}{n+m}$

$(s,i) \to (s, i-1)$ $\quad \gamma i$

Sellke construction for stochastic SIR model

- works for non-Markovian case

Define an infection pressure:

$$\Phi(t) = \frac{\lambda}{n+m} \int_0^t I(s)\,ds$$

Each initially susceptible person (label them $1, \dots, n$) has an infection threshhold $Q_j$ which is random. In particular, the $Q_j$ are i.i.d. rv's with pdf

$$p_Q(z) = e^{-z}$$

Susceptible $\# j$ becomes infected at the time $t$ when $\Phi(t) = Q_j$

Each person, when infected, remains infective for random time $T$.

Let $Q_{(j)}$ be the threshhold of the $j$th weakest susceptible. (order statistics)

Why is this equivalent to the stochastic SIR model,

Recovery correct so just need to check infection process

Need to show that, under the Sellke construction, the probability that a given susceptible person becomes infected during time interval $[t, t+\Delta t]$ with probability $\frac{\lambda}{n+m} I(t)\Delta t + o(\Delta t)$

higher order

Check: According to Sellke construction,

Prob(Susceptible #j becomes infected during $[t, t+\Delta t]$ | j still susceptible at time t)

$$= \text{Prob}\left(\Phi(t+\Delta t) > Q_j \mid \Phi(t) < Q_j\right)$$

$$= 1 - \text{Prob}\left(Q_j > \Phi(t+\Delta t) - \Phi(t)\right)$$ because an exp. dist. variable $Q$ has the property $\text{Prob}(Q > b \mid Q > a) = \text{Prob}(Q > b - a)$

$$= 1 - e^{-(\Phi(t+\Delta t) - \Phi(t))}$$

$$\doteq 1 - (1 - \Phi'(t)\,\Delta t + o(\Delta t))$$

$$= \Phi'(t)\,\Delta t + o(\Delta t)$$

$$= \frac{\lambda}{n+m} I(t)\,\Delta t + o(\Delta t)$$

Key question: How many people are infected before the epidemic dies out? Let $Z$ be this random variable.

$$Z = \min\left\{ i : Q_{(i+1)} > \frac{\lambda}{n} \sum_{j=-(m-1)}^{i} T_j \right\}$$

~~[illegible]~~ $T_j$ are the infective times for person # $(j)$

$(j) \in 1, \dots, n$ : initial susceptibles

$(j) \in -(m-1), \dots, 0$ : initial infected

How do we calculate statistics of $Z$?

Clever recursive argument.

Relate epidemic statistics for a given population to those of a population with a smaller # of initial susceptibles.

Let $Z^{n'}$, $\Phi^{n'}$ be the total epidemic size and infection pressure with $n'$ initial susceptibles

<u>but</u> always normalize infection rate by $\frac{1}{n+m}$

↑ not $n'$

$$\Phi^{n'} = \frac{\lambda}{n+m} \int_0^t I^{n'}(s)\, ds$$

$$S^{n'}(0) = n'$$