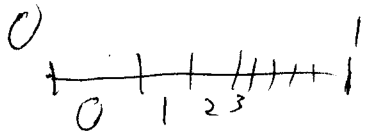
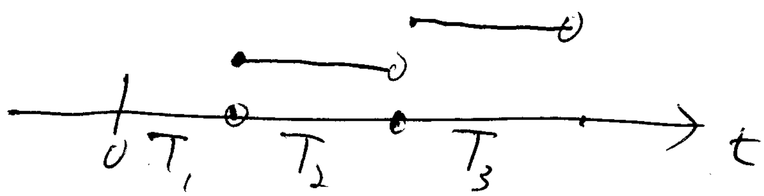
04/01/04

Simulating Poisson rv's for Poisson point process

$$\mathrm{Prob}(Y=n) = e^{-\lambda}\frac{\lambda^n}{n!} \quad \text{for } n \in \mathbb{Z}_{\geq 0}$$



Recall that for the Poisson counting process $X(t)$ with transition rate λ:



$\mathrm{Prob}(T_j > t) = e^{-\lambda t}$ and the T_j are i.i.d.

$$\mathrm{Prob}(X(1)=n) = e^{-\lambda}\frac{\lambda^n}{n!}$$

So to simulate a Poisson rv with mean λ:

generate iid exp dist rv's with pdf

$$p_T(t) = \lambda e^{-\lambda t}$$

Treat each of these as a T_j (time to next event.)

Keep simulating until $\sum_{j=1}^{n} T_j > 1$

Then set $Y_{sim} = n-1$ and this simulated rv will satisfy $\mathrm{Prob}(Y=j) = e^{-\lambda}\frac{\lambda^j}{j!}$

Continuous-time Markov chains:

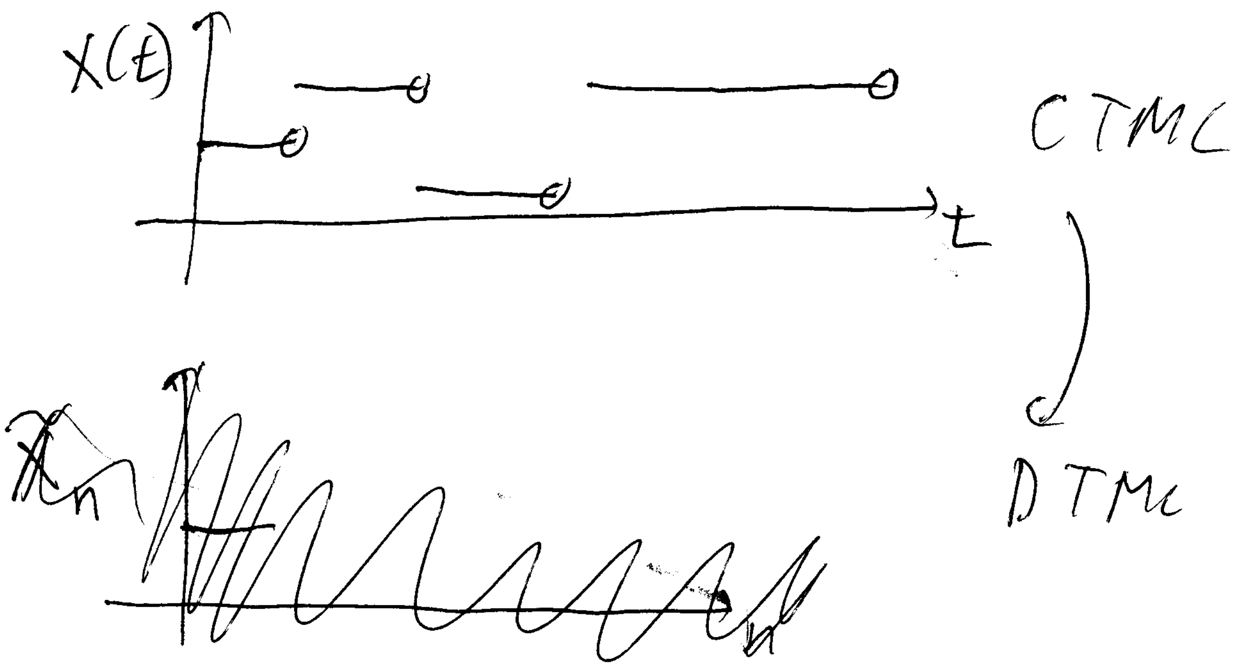
Long-time properties

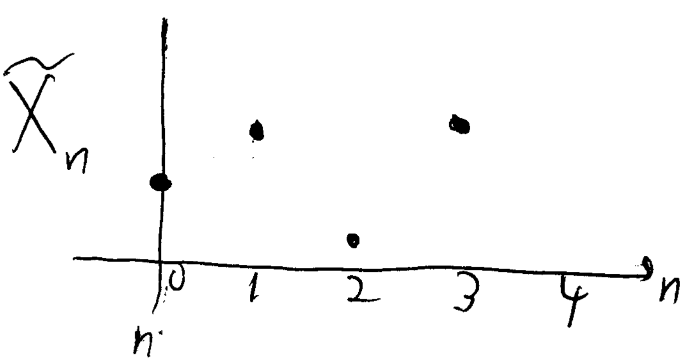
Some can be done using discrete-time MC techniques by associating to a continuous-time MC with inf. gen A a discrete-time MC with prob. trans. matrix

$$\tilde{P}_{ij} = \frac{A_{ij}}{\bar{A}_i} \quad \text{for } i \neq j \qquad \hat{P}_{ii} = 0$$

except for absorbing states k: set $\tilde{p}_{kj} = \delta_{kj}$.

This associated DTMC has each epoch correspond to 1 transition of CTMC





; keeps track of states visited and in what order.

Note: $\hat{X}_n \neq \tilde{X}_{n+1}$

Transience, recurrence, absorption probs for CTMC $X(t)$ can be decided by looking at associated DTMC $\hat{X}_n$.

But properties positive recurrence and expected time to absorption + stationary dist require info about time spent in a state so can't answer from assoc. DTMC.

Lawler Ch. 3

Resnick Sec. 5.5

Stationary distribution $\vec{\pi}$ for CTMC.

$$\pi_i = \text{Prob}(X(t) = i)$$

$$\frac{d\vec{\pi}}{dt} = 0 \qquad \text{Kolmogorov forward: } \frac{d\vec{\pi}}{dt} = \vec{\pi} A$$

$$\boxed{\vec{\pi} A = 0}$$

$$\boxed{\sum_{j \in S} \pi_j = 1}$$

: not the same stat dist as that of the associated DTMC

This equation has similar interpretation as discrete case, ~~but now~~ ~~total~~ ~~flu~~ total flux in = total flux out:

$$\sum_{j \neq i} \pi_j A_{ji} = \pi_i \bar{A}_i = \sum_{j \neq i} \pi_i A_{ij}$$

Can often used detailed balance ideas. Haken

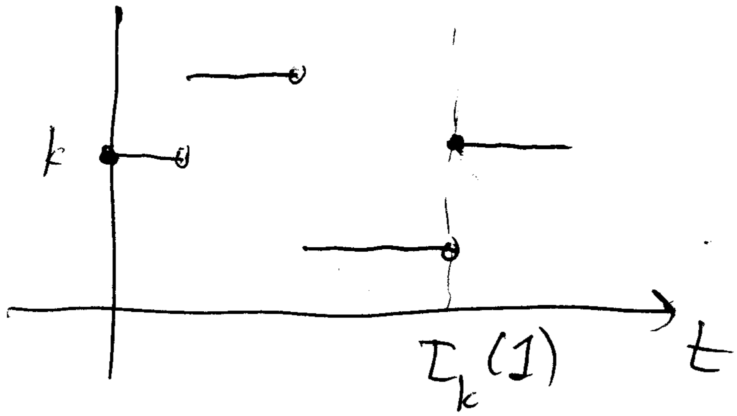
Positive recurrence for CTMC:

when recurrent, construct invariant measure $\vec{n}$ for CTMC by fix ref state k and letting $X(0)=k$ and

$$n_j = \mathbb{E} \int_0^{\tau_k(1)} I\{X(s)=j\}\, ds$$

= expected ~~time~~ spent in state j between visits to k.

$$\tau_k(1) = \min\{t \geq 0 : X(t)=k \text{ and } X(t-\varepsilon) \neq k \text{ for some } 0<\varepsilon<t\}$$



Show: $\vec{n} A = 0$ (invariant measure)

Note: $\sum_{j \in S} n_j = \mathbb{E}\, \tau_k(1)$

It is true that: $\mathbb{E}\, \tau_k(1) < \infty \iff$ positive recurrent

$\iff$ existence of stat dist $\vec{\pi} = \dfrac{\vec{n}}{\sum_{j \in S} n_j}$

$\iff \vec{\pi}$ is the limit dist of CTMC.

Expected cost / reward until absorption into a class or state.

~~Let $f: I \to \mathbb{R}$ (function defined~~ on ~~transient sta~~

Let f be a function on the state space S of the CTMC $X(t)$.

Want to calculate: $\mathbb{E}\left[\int_0^T f(X(t))\,dt \,\middle|\, X(0)=i\right]$

where T is the first time $X(t)$ enters a certain state k (or class of states).

f is a <u>rate</u> of cost or reward.

Could do this by first step analysis based on first transition time.

But we will instead do it by using Kolmogorov backward eqn which makes our argument generalize to the case where $X(t)$ ~~is a~~ varies continuously (SDE).

To warm up, consider first:

$$w_i(t) = \mathbb{E}\left[\int_0^t f(X(s))\,ds \,\middle|\, X(0)=i \right]$$

Upper time limit t is deterministic

Kolmogorov backward eqn says:

$$u_i(t) = \mathbb{E}\left[f(X(t)) \,\middle|\, X(0)=i \right]$$

satisfies

$$\frac{d\vec{u}}{dt} = A\vec{u}$$

$$\vec{u}(t=0) = \vec{f}$$

Formal solution: $\vec{u}(t) = e^{At}\vec{f}$

- literally a solution if A is finite-dim matrix.
- if A is infinite-dim think of this as a notation for the solution of the eqn.
  - semigroup theory

$$w_i(t) = \int_0^t \mathbb{E}\left[f(X(s)) \,\middle|\, X(0)=i \right] ds$$

$$= \int_0^t u_i(s)\,ds$$

$$\vec{w}(t) = \int_0^t \vec{u}(s)\,ds = \int_0^t e^{As}\vec{f}\,ds$$

$$= A^{-1}\left(e^{At} - I\right)\vec{f}$$

could do L'Hopital's rule for matrices

A^{-1} doesn't really exist (A may have 0 ev).

More rigorous + simpler:

$$A\vec{w}(t) = \int_0^t Ae^{As}\vec{f}\,ds = \left[\int_0^t \frac{d}{ds}\left(e^{As}\right)ds\right]\vec{f}$$

$$A\vec{w}(t) = \left(e^{At} - I\right)\vec{f}$$

Note that $\vec{w}'(t) = e^{At}\vec{f}$

So get autonomous DE for $\vec{w}(t)$:

$$\boxed{\begin{aligned} \frac{d\vec{w}}{dt} &= A\vec{w} + \vec{f} \qquad \text{for } t > 0 \\ \vec{w}(t=0) &= \vec{0} \end{aligned}}$$

where $w_i(t) = \mathbb{E}\left[\int_0^t f(X(s))\,ds \,\middle|\, X(0) = i\right]$

This formula generalizes to case of SDE's.

Now consider

$$V_i = \mathbb{E}\left[\int_0^{T_k(1)} f(X(t))\,dt \,\middle|\, X(0) = i\right]$$

where $T_k(1)$ = first hitting time of state k,

~~To do this assume f~~

Assume f is bounded and that $\text{Prob}(T_k(1) < \infty) = 1$

We will set ~~f(k)=0~~ and make k an absorbing state.

$\hat{f}(i) = f(i)$ for $i \neq k$

$\hat{f}(k) = 0$

The modified inf. gen w/ k a trapping state is defined:

$$\hat{A}_{ij} = A_{ij} \quad \text{for } i \neq k$$

$$\hat{A}_{kj} = 0$$

$\hat{X}(t)$ is the CTMC ~~as~~ w/ inf gen $\hat{A}$

Then $$\int_0^{\tau_k(1)} f(X(t))\,dt = \int_0^{\tau_k(1)} \hat{f}(\hat{X}(t))\,dt$$
$$= \int_0^{\infty} \hat{f}(\hat{X}(t))\,dt$$

Apply prev result to last expression:

$$\hat{A}\vec{v} = \lim_{t\to\infty} \left(e^{\hat{A}t} - I\right)\vec{\hat{f}}$$

$$\lim_{t\to\infty} e^{\hat{A}t}\vec{\hat{f}} = \lim_{t\to\infty} \underset{\substack{\downarrow \\ \text{trans. prob} \\ \text{for } \hat{X}}}{\hat{P}(t)}\,\vec{\hat{f}} = 0 \text{ w/ prob. } 1$$

because $$\left|\sum_{j\in S} \hat{P}_{ij}(t)\hat{f}_j\right| = \left|\sum_{j\neq k} \hat{P}_{ij}(t)\hat{f}_j\right|$$
$$\leq \|f\|_\infty \sum_{j\neq k} \hat{P}_{ij}(t) \to 0 \text{ w/ prob. } 1, \text{ since}$$
$\tau_k(1) < \infty$ w/ prob. 1.

So $\hat{A}\vec{v} = -\vec{\hat{f}}$

Rewrite this as:

$$\boxed{\begin{gathered}(A\vec{v})_j = -f(j) \quad \text{for } j \neq k \\ \text{with } \quad v(k) = 0\end{gathered}}$$

This generalizes to SDE.

Here $v_i = \mathbb{E}\left[\int_0^{T_k(1)} f(X(t))\,dt \,\middle|\, X(0)=i\right]$

This generalizes directly ~~to~~ for calculating $\mathbb{E}\left[\int_0^T f(X(t))\,dt \,\middle|\, X(0)=i\right]$

for T = first hitting time of class of states.

Or even for *any* Markov time T.

- modify MC by introducing another state that is entered when the Markov time is reached.

(Converts Markov time T to first hitting time of a state in extended MC)

Optimal stopping theory: $\boxed{\text{Lawler Ch. 4}}$: dynamic programming

- tries to find *optimal* Markov time T

Examples for long time properties of CTMC:

1) Birth-death processes: State space $S = \mathbb{Z}_{\geq 0}$.

$$A_{i,i+1} = \lambda_i \quad \text{for } i \geq 0$$

$$A_{i,i-1} = \mu_i \quad \text{for } i \geq 1$$

$$A_{i,i} = -\bar{A}_i = -(\lambda_i + \mu_i)$$

$$A_{i,j} = 0 \quad \text{for } |i-j| > 1$$

Invariant measure $\vec{n}$:

$$\vec{n} A = 0.$$

Same kind of calculation as for DTMC birth-death chain.

Detailed balance works:

$$n_j = \prod_{i=0}^{j-1} \frac{\lambda_i}{\mu_{i+1}}$$

Stat dist exists if $\sum_{j=0}^{\infty} n_j < \infty$

$$\vec{\pi} = \frac{\vec{n}}{\sum_{j=0}^{\infty} n_j}.$$

Can study transience in same way (Lawler Sec. 3.3)

Consider a special case of birth-death process consisting of single-server queue

$\lambda_i = \lambda$ for $i \geq 0$ (rate of arrival of demand)

$\mu_i = \mu$ for $i \geq 0$ (rate of completing service)

$$n_j = \left(\frac{\lambda}{\mu}\right)^j \qquad \sum_{j=0}^{\infty} n_j = \frac{1}{1-\frac{\lambda}{\mu}} \quad \text{if } \lambda < \mu$$

$$\pi_j = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^j$$: geo dist. for $\lambda < \mu$

If $\lambda > \mu$: transient

$\lambda = \mu$: null recurrent

Focus on positive recurrent case $\lambda < \mu$:

What is the avg # requests waiting

$$\langle X(t) \rangle = \sum_{j \in S} j \pi_j \quad \text{for large } t$$

$$= \sum_{j=0}^{\infty} j \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^j$$

$$= \frac{\lambda}{\mu - \lambda}$$

using $\displaystyle\sum_{j=0}^{\infty} j \alpha^j = \alpha \frac{d}{d\alpha} \sum_{j=0}^{\infty} \alpha^j = \alpha \frac{d}{d\alpha} \frac{1}{1-\alpha}$

$$= \frac{\alpha}{(1-\alpha)^2}$$