03/29/04

HW3 due date now April 9 at 5PM (Friday)

Poisson counting process

- $X(t)$ increments by 1 at random times.

The time T spent in a state is a rv with prob density

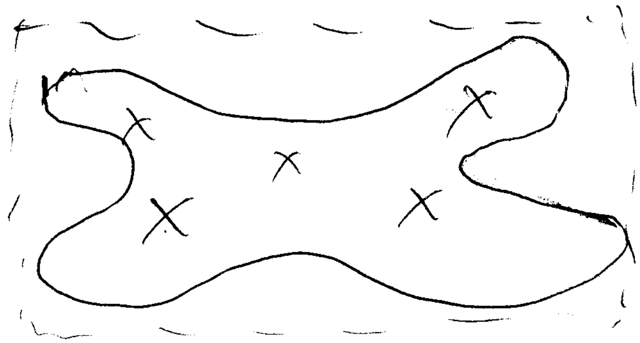
$$p_T(t) = \lambda e^{-\lambda t}$$ where λ = rate

If T had a different prob. dist., $X(t)$ would not be Markov process

→ renewal theory (Karlin + Taylor Ch. 5)

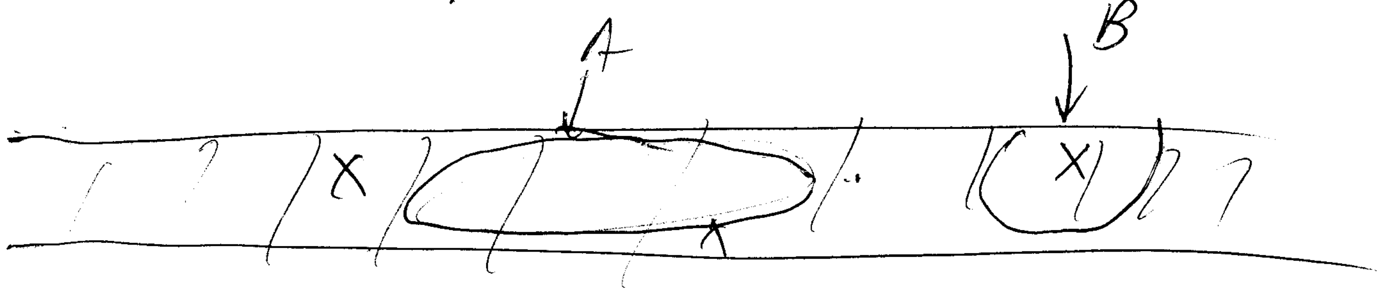
This Poisson counting process is related to <u>Poisson point process</u>, which is a way to distribute points in a uniform random way on an unbounded domain.

Uniform distribution of fixed # points on bounded domain



Use rejection method by generating uniform distribution on a box containing the domain

But what if domain unbounded, like the set of future times or an extended piece of material w/ defects



Examples: defects in material
times of radioactive decay
vortices in a fluid
distribution of noninteracting population
arrival of service requests

Poisson point process has a state space D^∞ where D is the domain over which points are distributed.

More concretely it can be characterized as a random ~~set~~ measure on subsets of D, taking values in $\mathbb{Z}_{\geq 0}$.

Let $N(A)$ be this random measure for subsets $A \subseteq D$.

- represents # Poisson points in A.

Poisson point process is uniquely defined by the properties

1) $N(A)$ and $N(B)$ are independent rv's when $A, B \subseteq D$ with $A \cap B = \emptyset$

2) $N(A)$ is a Poisson rv with mean $\lambda \, Vol(A)$ where λ is the <u>intensity</u> of the Poisson point process

Poisson rv X with mean μ:

$$Prob(X = n) = e^{-\mu} \frac{\mu^n}{n!} \quad \text{for } n \in \mathbb{Z}_{\geq 0}.$$

Why is the Poisson distribution for $N(A)$ a good one?

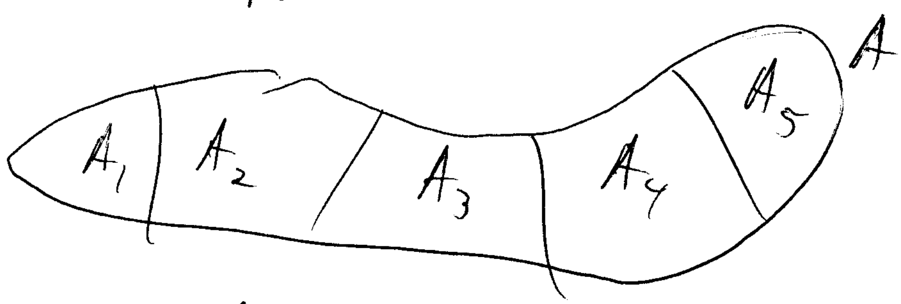
Poisson random variables have the special props:

1) They're integer-valued
2) They have infinitely divisible property.

Let $X^{(\mu)}$ be a Poisson rv with mean μ.

$$X^{(\mu)} \sim \sum_{j=1}^{n} X_j^{(\mu/n)} \quad \text{for any } n \in \mathbb{Z}_{\geq 0}$$

has same PDF   i.i.d.



Why does Poisson rv have infinitely divisible property?

Consider prob generating function $\ _{X^{(\mu)}}(s)$

for $X^{(\mu)}$: $G_{X^{(\mu)}}(s) = \langle e^{X^{(\mu)}} \rangle$

Infinite divisible $\Rightarrow$ $G_{X^{(\mu)}}(s) = \left(G_{X^{(\mu/n)}}(s)\right)^n$

for any $n \in \mathbb{Z}_{\geq 0}$

$\Rightarrow$ $G_{X^{(\mu)}}(s) = \left(G_{X^{(1)}}(s)\right)^{\mu}$

for any $\mu \geq 0$

(approx μ by rationals, use monotone dependence)

So in particular $\left(G_{X^{(\mu)}}(s)\right)^{1/\mu}$ must be a prob. gen. fn. for any μ.

For Poisson rv:

$$P_{X^{(\mu)}}(s) = \sum_{n=0}^{\infty} e^{-\mu} \frac{\mu^n}{n!} s^n = e^{-\mu} e^{\mu s} = e^{\mu(s-1)}$$

So we see that for any μ,

$$\left(P_{X^{(\mu)}}(s)\right)^{1/\mu} = e^{(s-1)} = P_{X^{(1)}}(s)$$

Most general infinitely divisible random variable which is integer-valued is: compound Poisson rv.

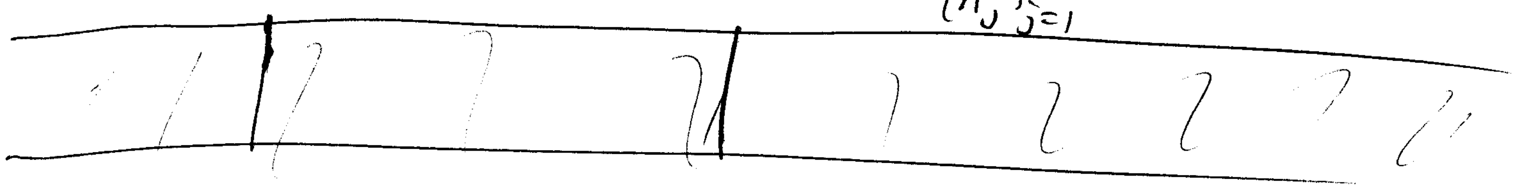
- each point would have some integer associated to it.

Infinitely divisible random variables w/
continuous state space
- Gaussian
- Compound Poisson rv's

General Theory: <u>Levy-Khinchine theorem</u>
Stroock, <u>Analytic Probability Theory</u>

Poisson counting process $X(t)$ is just the # points generated by a Poisson point process in interval $[0, t]$. $\text{Prob}(X(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$ by solving Kolmogorov forward eqn.

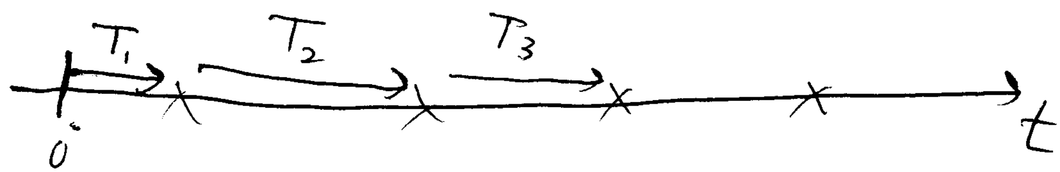
Numerical simulation of Poisson point process
- focus on one (or a small number) of bounded domains of interest $\{A_j\}_{j=1}^n$



For each domain A_j, calculate $N(A_j)$ as a Poisson rv w/ mean $\lambda \, \text{Vol}(A_j)$

Given that a Poisson point process places m points in a region A_j, the m points are uniformly distributed in A_j.

For one dimension, can also simulate Poisson point process by generating i.i.d random variables T_j with prob. density $p_{T_j}(t) = \lambda e^{-\lambda t}$. The T_j represent times between Poisson points



Strong Markov Property needed to explain simulation of CTMC.

Given a stochastic process $X(t)$, let $\mathcal{A}_t$ denote the σ-algebra of <u>events</u> which can be decided up to time t.

For separable (continuous) processes $X(t)$ w/ associated probability space, $\mathcal{A}_t$ is generated by sets

$$A = \{ \omega \in \Omega : X_{s_i}(\omega) \in B_i \text{ for } i = 1, \dots, n \}$$

where B_i are Lebesgue-measurable sets for $\{s_i\}_{i=1}^{n} \leq t$.

The σ-algebra family $\mathcal{A}_t$ is called a <u>filtration</u> if $\mathcal{A}_t \subseteq \mathcal{A}_{t'}$ for $t \leq t'$.

Markov property:

$$\text{Prob}(X(t) = j \mid \mathcal{A}_s) = \text{Prob}(X(t) = j \mid X(s))$$

for $s < t$

What about if the past and future are separated by a random time, like time of transition in a CTMC?

Markov time (stopping time) τ is a random variable defined on the prob. space Ω such that

$$\{\tau \leq t\} \in \mathcal{A}_t \text{ for all } t \geq 0.$$

event

This means you can tell whether the event associated to τ happened by a certain time by only knowing past history up to that point.

Examples:

- time at which one changes state in a CTMC
- first hitting times
- second hitting time
- time to accumulate a certain reward
- but not last hitting time.

Strong Markov property:

For Markov times τ

$$\text{Prob}(X(t+\tau)=j \mid \mathcal{A}_\tau) = \text{Prob}(X(t+\tau)=j \mid X(\tau))$$

- generalize Markov property (future predictions, given the present, are independent of past) to the case where the present time is a random Markov time.

Continuous-time MC has strong Markov property if right continuous.

Friedman, <u>Stochastic Differential Equations</u>

Numerical simulation of continuous-time MC
- discretizing by fixed time increments Δt can be inefficient because most of the time no transitions happen

A more efficient approach:
Start in state $X(0) = i$.

Generate random variable T until a transition happens

$$\text{Prob}(T > t \mid X(0) = i) = e^{-\bar{A}_i t}$$

where $\bar{A}_i = \sum_{j \neq i} A_{ij}$ is total rate of leaving state i.

Exponential rv's easy to simulate (inverse transform method)

Where does it go when the MC makes a transition.

$$\text{Prob}(X_+ = j \mid X_- = i) = \frac{A_{ij}}{\bar{A}_i}$$

Then repeat process from new state.
Rigorous justification: Karlin + Taylor Ch. 14

This method is equivalent to having an alarm clock associated to each transition $i \to j$. These alarm clocks are set when enter state i, and ring at a time T_j later.

$$\text{Prob}(T_j > t) = e^{-A_{ij} t}, \quad \text{independent of each other.}$$

The first one to ring determines the time and target of the transition from i.

Why are these equivalent?

Let $T = \min_j T_j$

$$\text{Prob}(T > t) = \text{Prob}(T_j > t \text{ for all } j) = \prod_{j \neq i} e^{-A_{ij} t}$$

$$= e^{-\sum_{j \neq i} A_{ij} t} = e^{-\bar{A}_i t}$$

$$\text{Prob}(X_+ = j \mid X_- = i) = \text{Prob}(T_j < T_{j'} \text{ for } j \neq j')$$

$$= \int_0^\infty dt \; p_{T_j}(t) \; \text{Prob}(\underset{\text{for } j' \neq j}{T_{j'} > T_j} \mid T_j = t)$$

$$= \int_0^\infty dt \; A_{ij} e^{-A_{ij} t} \prod_{j' \neq j, i} e^{-A_{ij'} t}$$

$$= A_{ij} / \bar{A}_i$$