03/25/04 Continuous-Time Markov Chains

Examples

Lawler Ch. 3
Karlin + Taylor Ch. 4

1) Birth-death processes where A is tridiagonal

$A_{i,i+1} = \lambda_i$ = aggregate birth rate

$A_{i,i-1} = \mu_i$ = aggregate death rate

$A_{i,i} = -\mu_i - \lambda_i$ (row sums = 0)

$A_{i,j} = 0$ for $|i-j| \geq 2$

a) Poisson process (# counts/arrivals)

$\lambda_i = \lambda, \quad \mu_i = 0$

b) Queueing theory with k servers

$\lambda_i = \lambda$ (arrival rate)

$\mu_i = i\mu$

$\mu_i = k\mu$ for $i \geq k$

where $\mu = \frac{1}{T_{service}}$ → average time for service

c) Population models

$\lambda_i = i\lambda$ or $\lambda_i = i^2\lambda$ or $\lambda_i = \lambda i(k-i)$

$\mu_i = i\mu$ or $\mu_i = i^2\lambda$ or $\lambda_i = i\lambda + r$

immigrants

Annoying technicality: continuous-time MC is not necessarily well-defined by the matrix A if state space is infinite.

- no problem if A is bounded
- have to possibly worry if A has unbounded entries (K+T p 135)
  - may be problems if $\mu_i \propto i^2$ and $\lambda_i \propto i^2$ and $\frac{\lambda_i}{\mu_i} \to 0$ as $i \to \infty$

2) Finite-state processes w/o linear ordering

- transitions between energy levels in QM
- machine switching between modes of operation
- biomolecule switch between conformational states

How do we compute statistical properties of the continuous-time MC from its infinitesimal generator A?

First we'll look at $P_{ij}(t) = \text{Prob}(X(t+t')=j \mid X(t')=i)$

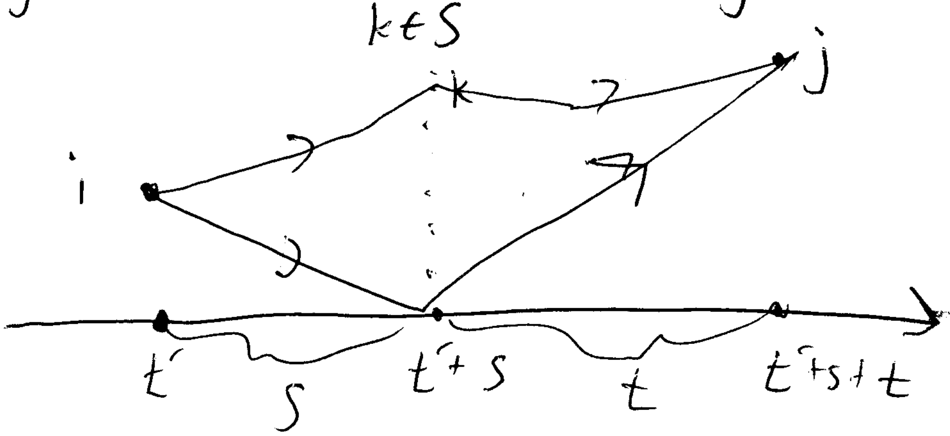
Recall $A = \left.\frac{dP}{dt}\right|_{t=0^+}$

If A exists, then
$$P_{ij}(\Delta t) = \delta_{ij} + \Delta t \, A_{ij} + o_{ij}(\Delta t)$$

higher order terms
$\frac{o_{ij}(\Delta t)}{\Delta t} \to 0$ as $\Delta t \to 0$

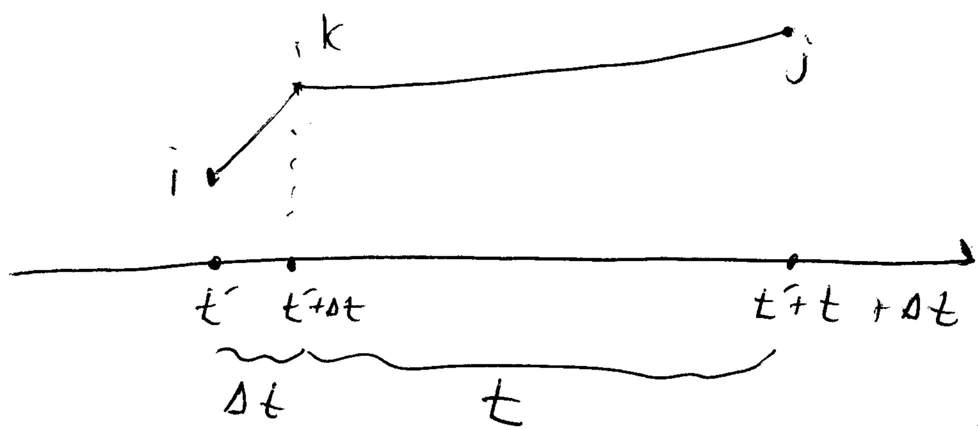
What about $P_{ij}(t)$ when t is not small?

Chapman-Kolmogorov eqn in continuous-time:
$$P_{ij}(t+s) = \sum_{k \in S} P_{ik}(s) P_{kj}(t)$$



We'll use Chapman-Kolmogorov to recursively extend the time interval over which we know $P_{ij}(t)$.

$$P_{ij}(t+\Delta t) = \sum_{k\in S} P_{ik}(\Delta t)\, P_{kj}(t)$$



$$= \sum_{k\in S} \left(\delta_{ik} + \Delta t\, A_{ik} + o_{ik}(\Delta t)\right) P_{kj}(t)$$

$$P_{ij}(t+\Delta t) = P_{ij}(t) + \Delta t \sum_{k\in S} A_{ik} P_{kj}(t) + \sum_{k\in S} o_{ik}(\Delta t) P_{kj}(t)$$

$$\frac{P_{ij}(t+\Delta t) - P_{ij}(t)}{\Delta t} = \sum_{k\in S} A_{ik} P_{kj}(t) + \sum_{k\in S} \frac{o_{ik}(\Delta t)}{\Delta t} P_{kj}(t)$$

Formal $\Delta t \to 0$ limit:

$$\boxed{\begin{aligned} \frac{dP_{ij}(t)}{dt} &= \sum_{k \in S} A_{ik} P_{kj}(t) \\ \frac{dP}{dt} &= AP \\ P(t=0) &= I \end{aligned}}$$

Backward Kolmogorov equation

Rigorous handling of the $\Delta t \to 0$ limit:

- use dominated convergence theorem with $P_{kj}(t) \leq 1$ and $\sum_{k \in S} \upsilon_{ik}(\Delta t) = \bar{\upsilon}_i(\Delta t) < \infty$
- Karlin + Taylor Vol. I + II

One could also repeat the argument starting from

$$P_{ij}(t) = \sum_{k \in S} P_{ik}(t) P_{kj}(\Delta t)$$

One gets from the formal calculation

$$\boxed{\begin{aligned} \frac{dP_{ij}(t)}{dt} &= \sum_{k \in S} P_{ik}(t) A_{kj} \\ \frac{dP}{dt} &= PA \\ P(t=0) &= I \end{aligned}}$$

Forward Kolmogorov Equation

The forward equation does not always make rigorous sense!

The solution to these equations (when they make sense) can be formally expressed as

$$P(t) = e^{At}.$$

Makes perfect sense for finite state space

Several papers in SIAM Review

"Nineteen Ways to Compute Matrix Exponentials"

Clive Moler and...

⊚ Forward vs. Backward Equation

A) Consider first $\phi_j(t) = \text{Prob}(X(t)=j)$

$$\phi_j(t) = \text{Prob}(X(t)=j) = \sum_{k\in S} \text{Prob}(X(t)=j \text{ and } X(0)=k)$$

$$= \sum_{k\in S} \text{Prob}(X(t)=j \mid X(0)=k)\,\text{Prob}(X(0)=k)$$

$$\phi_j(t) = \sum_{k\in S} P_{kj}(t)\,\phi_k(0)$$

$$\vec{\phi}(t) = \vec{\phi}(0)\cdot P(t)$$

vector ↑ ↑ matrix

Can avoid solving eqn for large matrix P by noting that $\vec{\phi}$ itself satisfies Kolmogorov forward eqn:

$$\boxed{\frac{d\vec{\phi}}{dt} = \vec{\phi}\cdot A} \qquad \left(= \underbrace{\vec{\phi}(0)\cdot P(t)}\cdot A\right)$$

but <u>not</u> Kolmogorov backward eqn.

What is the Kolmogorov backward eqn good for?

Expectations starting from given state.

Let $u_i(t) = \langle f(X(t)) \mid X(0) = i \rangle$

$$= \sum_{j \in S} f(j)\, \text{Prob}(X(t) = j \mid X(0) = i)$$

$$= \sum_{j \in S} f(j)\, P_{ij}(t)$$

$$\vec{u}(t) = P(t) \cdot \vec{f}$$

This satisfies Kolmogorov backward equation:

$$\boxed{\begin{aligned} \frac{d\vec{u}}{dt} &= A\vec{u} \\ \vec{u}(t=0) &= \vec{f} \end{aligned}}$$

but <u>not</u> the forward eqn.

Forward equation describes how <u>probabilities</u> at a <u>future</u> time evolve.

Backward equation describes how <u>expected "payoffs"</u> evolve given an <u>initial</u> state.

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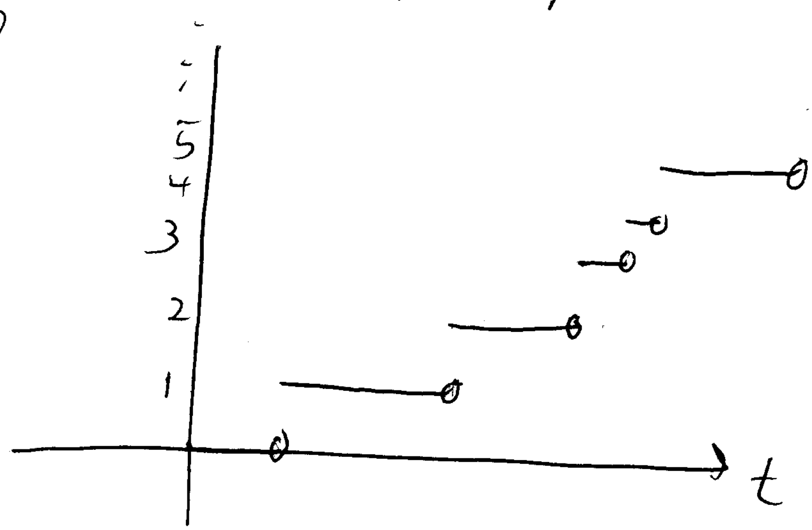
Poisson process

Poisson *counting* process $X(t)$ is a birth-death process with $X(0)=0$, $\lambda_i = \lambda$, $\mu_i = 0$.

Infinitesmal generator

$$A = \begin{pmatrix} -\lambda & \lambda & & & 0 \\ 0 & -\lambda & \lambda & & \\ & 0 & -\lambda & \lambda & \\ & & 0 & \ddots & \ddots \\ 0 & & & & \ddots \end{pmatrix}$$

What is the probability distribution for the amount of time $X(t)$ spends in a given state?



Consider $\phi_0(t) = \text{Prob}(X(t)=0)$

Initialize : $\phi_0(0) = \underline{1}$.

$$\frac{d\vec{\phi}}{dt} = \vec{\phi} \cdot A$$ Kolmogorov forward eqn

0th component: $$\frac{d\phi_0}{dt} = -\lambda \phi_0$$

$$\phi_0(0) = 1$$

$$\phi_0(t) = e^{-\lambda t}$$

Let T be the random time at which $X(t)$ leaves state 0:

$$T = \inf_{t \geq 0} \{ X(t) \neq 0 \}$$

$$\text{Prob}(T > t) = \phi_0(t) = \text{Prob}(X(t) = 0)$$

$$= e^{-\lambda t}$$

So if we write $\text{Prob}(T \in B) = \int_B p_T(t)\,dt$

then $p_T(t) = \lambda e^{-\lambda t}$

The transition time in a Poisson process has exponential distribution.

- same for any state

There is a deeper reason for why the time ~~between~~ to transition, T, has exponential distribution.

- Markov property that determines it.

$$\text{Prob}(T > t \mid T > s) = \text{Prob}(T > t - s)$$

for $t > s$.

because given that $T > s$, there is no memory about what happened before time s.

$$\text{Prob}(X(t) = 0 \mid X(s) = 0, X(0) = 0)$$

$$\text{Prob}(X(t) = 0 \mid X(s) = 0)$$

By defn of cond. prob

$$\frac{\text{Prob}(T > t)}{\text{Prob}(T > s)} = \text{Prob}(T > t - s)$$

Let $G(t) = \text{Prob}(T > t)$

$$*) \quad \frac{G(t)}{G(s)} = G(t-s)$$

Clearly one solution is $G(t) = e^{-\lambda t}$ for some $\lambda > 0$.

This is the only class of solutions.

If $G(t)$ is differentiable, then

if differentiate $*$) with respect to s,

and write $u = t - s$:

$$G(t) = G(t-s)\,G(s)$$

$$0 = -G'(u)\,G(s) + G(u)\,G'(s)$$

$$\frac{G'(s)}{G(s)} = \frac{G'(u)}{G(u)} \quad \text{for all } s, u \geq 0.$$

$$\therefore \frac{G'(s)}{G(s)} = \text{constant}$$

There are also no nonsmooth solution

Karlin + Taylor Th. 4.2.2