03/22/04 Calculations w/ branching processes

Means?

$$\langle X_n \rangle = \frac{d}{ds} P_{X,n}(s)\Big|_{s=1} = \frac{d}{ds}\left(P_Y(P_{X,n-1}(s))\right)\Big|_{s=1}$$

$$= \left[P_Y'(P_{X,n-1}(1))\right] P_{X,n-1}'(1) \quad \text{chain rule}$$

$$= P_Y'(1)\, P_{X,n-1}'(1) = \mu \langle X_{n-1} \rangle$$

where $\mu = \langle Y \rangle$ = mean # descendants of 1 parent in next generation

$$\langle X_0 \rangle = 1$$

$$\langle X_n \rangle = \mu^n \quad \text{by induction}$$

Can calculate variances by second derivatives

Any finite time statistic about branching process can be obtained from

$$P_{X,n}(s) = P_Y^{\circ n}(s) \quad \text{for } X_0 = 1.$$

Can also derive an explicit formula when $X_0 \neq 1$ or even random.

Stochastic update rule:

$$X_{n+1} = \sum_{k=1}^{X_n} Y_{n,k} \qquad X_0 = 1$$

where $\{Y_{n,k}\}$ are iid rv representing # offspring of an agent in next generation (including parent if it survives)

This is a countable-state Markov chain

Because we're dealing with random sums with nice recursive structure, generating functions work well.

$$P_{X,n}(s) = \mathbb{E}\, s^{X_n} \equiv \langle s^{X_n} \rangle$$

$$P_Y(s) = \mathbb{E}\, s^Y$$: gen fn for # offspring of one parent.

$$P_{X,n+1}(s) = \mathbb{E}\, s^{X_{n+1}} = \mathbb{E}\, s^{\sum_{k=1}^{X_n} Y_{n,k}}$$
$$= P_{X,n}(P_Y(s))$$

Long time properties?

Probability of extinction

Let $a(k) = \text{Prob}(X_n = 0 \text{ for some } n < \infty \mid X_0 = k)$

Note: $a(k) = (a(1))^k$ (Each ancestor must lead to extinction independently)

First step analysis: Let $a = a(1)$

$$a = \text{Prob}\left(\bigcup_{n=1}^{\infty} \{X_n = 0\} \mid X_0 = 1\right)$$

$$= \sum_{k=0}^{\infty} \text{Prob}\left(\bigcup_{n=1}^{\infty} \{X_n = 0\} \text{ and } X_1 = k \mid X_0 = 1\right)$$

$$= \sum_{k=0}^{\infty} \text{Prob}\left(\bigcup_{n=1}^{\infty} \{X_n = 0\} \mid X_1 = k \text{ and } X_0 = 1\right) \times \text{Prob}(X_1 = k \mid X_0 = 1)$$

(conditional prob. law)

$$= \sum_{k=0}^{\infty} \text{Prob}\left(\bigcup_{n=1}^{\infty} \{X_n = 0\} \mid X_1 = k\right) \text{Prob}(X_1 = k \mid X_0 = 1)$$

Markov property

$$= \sum_{k=0}^{\infty} a(k)\, p_k \quad \text{where } p_k = \text{Prob}(Y = k)$$

$$a = \sum_{k=0}^{\infty} p_k a^k = \mathbb{E}\, a^Y = \mathcal{P}_Y(a)$$

Extinction probability for single ancestor satisfies

$$a = P_Y(a)$$

This equation may have multiple solutions... which one is right?

Boring cases:

A) $p_0 = 0, \quad p_1 = 1, \quad p_k = 0 \quad \text{for } k \geq 2$

$$X_n = X_0$$

B) $p_0 = p, \quad p_1 = 1 - p, \quad p_k = 0 \quad \text{for } k \geq 2$

$$a = 1$$

Focus on case where $p_k > 0$ for some $k \geq 2$.

$$0 \leq \underbrace{P_Y(0)}_{= p_0} \leq 1 \qquad\qquad P_Y(1) = 1$$

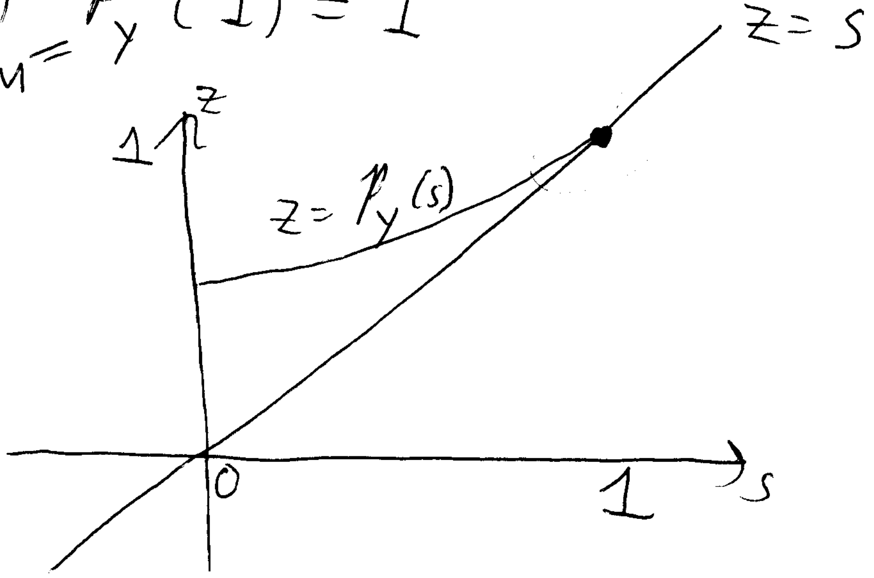
$$P_Y(s) = \sum_{k=0}^{\infty} p_k s^k$$

$$P_Y'(s) > 0 \quad \text{for } 0 < s \leq 1$$

$$P_Y''(s) > 0 \quad \text{for } 0 < s \leq 1$$

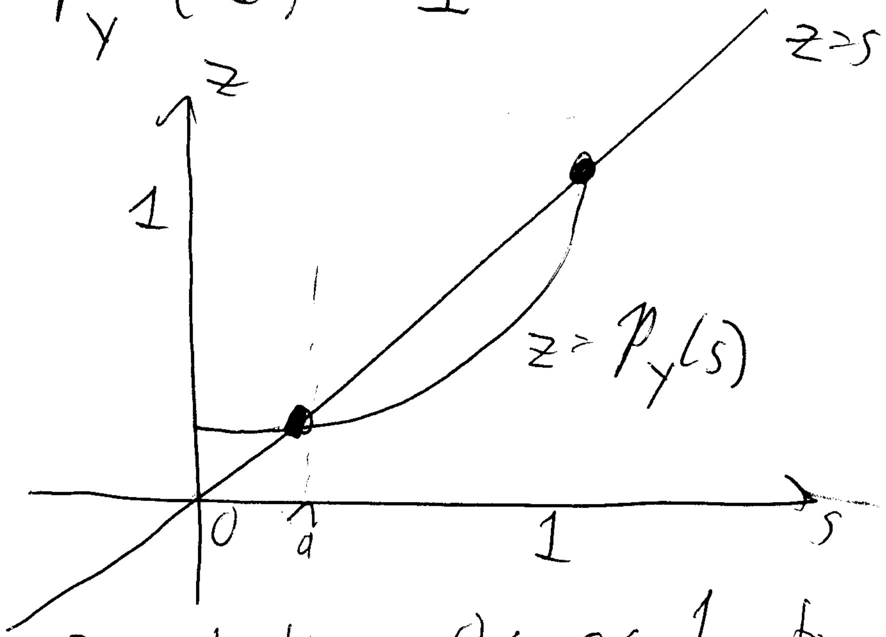
Two basic cases

I) $\underbrace{P_Y'(1)}_{\mu =} \le 1$



Only solution to $P_Y(s) = s$ for $0 \le s \le 1$ is $s = 1$

II) $\overset{\mu}{\overset{\parallel}{P_Y'(1)}} > 1$



2 solutions $0 \le s \le 1$ to $s = P_Y(s)$. Which is the correct value for a.

For case II), it is the smallest solution of $s = \mathcal{P}_Y(s)$ which gives the value for extinction probability a.

Reason:

Let $a_N = \text{Prob}(X_N = 0 \mid X_0 = 1)$

$$= \text{Prob}\left(\bigcup_{n=1}^{N} \{X_n = 0\} \mid X_0 = 1\right)$$

$$= \mathcal{P}_{X,N}(0) = \mathcal{P}_Y^{\circ N}(0)$$

Claim that $a_N \leq \hat{a}$ for all $N \geq 0$

where $\hat{a}$ is smallest solution to $\mathcal{P}_Y(s) = s$.

Proof of claim by induction:

$a_0 = 0$ ✓   Now assume $a_N \leq \hat{a}$

$$a_{N+1} = \mathcal{P}_Y(\mathcal{P}_Y^{\circ N}(0)) = \mathcal{P}_Y(a_N) \leq \mathcal{P}_Y(\hat{a}) = \hat{a}$$

by monotonicity of $\mathcal{P}_Y$

So $a_N \leq \hat{a} \implies a_{N+1} \leq \hat{a}$.

So by induction, $a_N \leq \hat{a}$ is true for all $N \geq 0$.

$a = \lim_{N\to\infty} a_N$ must be the smallest root $\hat{a}$.

Summary of long-time properties of branching processes. (μ = mean # offspring/parent

1) If $\mu \le 1$, and if $p_k \neq 0$ for some $k \ge 2$, then the population will go extinct w/ probability 1.

2) Boring case: If $\mu = 1$ and $p_0 = 0$, $p_1 = 1$, $p_k = 0$ for $k \ge 2$, then obviously $X_n = X_0$ for all time.

3) If $\mu > 1$, then the probability for extinction given $X_0 = k$ is $\hat{q}^k$ where $\hat{q}$ is the smallest nonnegative ~~positive~~ solution to $\hat{q} = P_Y(\hat{q})$.
$0 \le \hat{q} < 1$.

What happens to branching processes with $\mu > 1$ when they don't go extinct?
They grow unboundedly and visit each value for population size a finite number of times (with prob. 1)

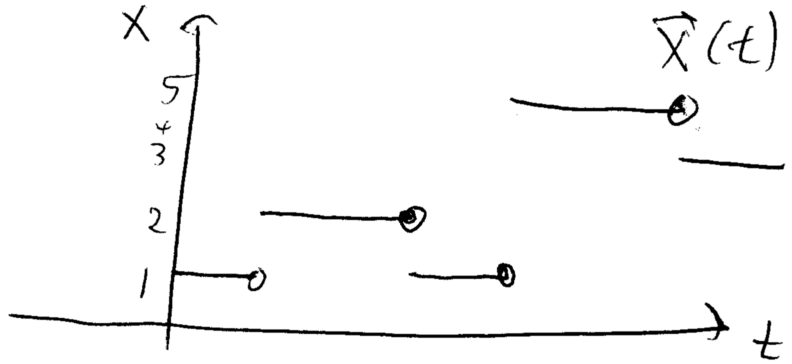
$$\lim_{n\to\infty} X_n = \infty$$

(MC is <u>transient</u>!)

Continuous-time Markov chains
(allow finite or countably many states)

The state of system now described by function $\vec{X}(t)$ where $-\infty < t < \infty$ need not be an integer: $t \in \mathbb{R}$

$\vec{X}(t)$ will jump between states, but at any times t.



Markov property:

$$\mathrm{Prob}(X(t)=j \mid X(t_1)=i_1, X(t_2)=i_2, \ldots, X(t_n)=i_n)$$

$$= \mathrm{Prob}(X(t)=j \mid X(t_n)=i_n)$$

with $t_1 < t_2 < \cdots < t_n < t$

Time-homogeneous MC:

$$\mathrm{Prob}(X(t)=j \mid X(s)=i) = P_{ij}(t-s) \quad \text{for } s \le t.$$

Discrete vs. Continuous Time Modeling

Note that making regular observations of continuous-time MC

$X_n = X(n \Delta t)$ makes X_n a discrete-time MC.

This means that if all I care about is state of process at regular time intervals, discrete-time MC may be more appropriate.

Why would we ever want to use continuous time MC rather than discrete time

1) Sensible for continuous space processes
   - work with differential equations
   - remove artefacts ~~the~~ from finite time step

2) Need it if you want to know if some profitable or dangerous state is ever achieved between some regular observation times

3) Continuous time MC may give simpler approx to discrete time dynamics

Mathematical formulation:

Models formulated in terms of transition <u>rules</u> rather than transition probabilities

Consider $P_{ij}(t) = \text{Prob}(X(t+t') = j \mid X(t') = i)$

(time-homogeneous)

is transition probability function

$$\lim_{t \to 0} P_{ij}(t) = \delta_{ij} = 1 \text{ if } i = j$$
$$= 0 \text{ if } i \neq j$$

$$\lim_{t \to 0} P(t) = I$$

Look at next correction term:

$$P(t) = I + At + o(t)$$

higher order terms

$$\frac{o(t)}{t} \to 0 \text{ as } t \to 0$$

A is the <u>infinitesmal generator</u> of the Markov process

$$A = \left.\frac{dP}{dt}\right|_{t=0^+}$$

For $i \neq j$, $$A_{ij} = \lim_{\Delta t \to 0} \frac{\text{Prob}(X(t+\Delta t) = j \mid X(t) = i)}{\Delta t} > 0$$

= rate of transition from $i \to j$

For $i = j$,
$$A_{ii} = -\lim_{\Delta t \to 0} \frac{\text{Prob}(X(t+\Delta t) \neq i \mid X(t) = i)}{\Delta t} < 0$$

= total rate of transition out of state i.

$$A_{ii} = -\sum_{j \neq i} A_{ij}$$

Equivalently: $$\sum_{j \in S} A_{ij} = \underline{\underline{0}}$$

To model a continuous time MC, need to specify off-diagonal components of the infinitesimal generator A
- $\underline{\text{rates}}$ of transitions between states

Dimensions of A is $\frac{1}{\text{time}}$

Off-diagonal entries: any positive number.