03/18/04

Examples of transience and recurrence in countable state MC.

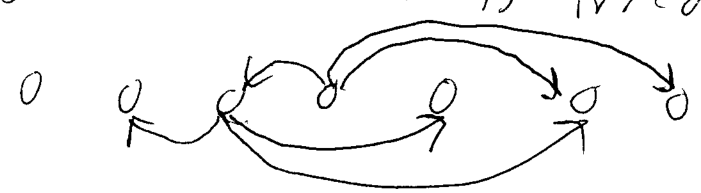
Example 1) Queueing model: Resnick 2.15
Lawler 2.3
Karlin + Taylor 3.5

X_n = # people in queue ~~state~~ right before $(n+1)$st service period

$$X_{n+1} = (X_n - 1 + \xi_n)_+$$

where ξ_n = # new requests during nth service period
- modeled as iid r.v.'s.

Result: Whole MC is irreducible



Consider $\rho = \langle \xi_n \rangle$ = average # arrivals per service period

If $\rho > 1 \Rightarrow$ transient (queue length $\to \infty$ w/ prob. 1 in limsup)

$\rho < 1 \Rightarrow$ positive recurrent
Prob$(X_n = j)$ has finite value π_j when n large

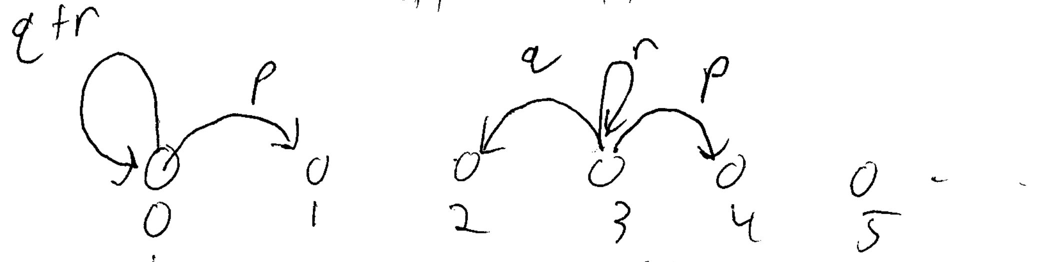
$\rho = 1 \Rightarrow$ null recurrent

Remark: The 1-D random walk:

$$\text{Prob}(X_{n+1} = X_n + 1) = p$$

$$\text{Prob}(X_{n+1} = X_n - 1) = q \qquad p+q+r = 1$$

$$\text{Prob}(X_{n+1} = X_n) = r$$



but with reflection at $j=0$

$$\text{Prob}(X_{n+1} = 0 \mid X_n = 0) = q + r$$

$$\text{Prob}(X_{n+1} = 1 \mid X_n = 0) = p$$

This is a special case of the queue model

$$\text{Prob}(\xi_n = 0) = q$$

$$\text{Prob}(\xi_n = 1) = r$$

$$\text{Prob}(\xi_n = 2) = p$$

$$\text{Prob}(\xi_n = j) = 0 \quad \text{for } j \geq 3$$

$$\rho = \langle \xi_n \rangle = r + 2p = 1 + p - q$$

Remark: Sometimes helpful also to think of queues in terms of branching processes

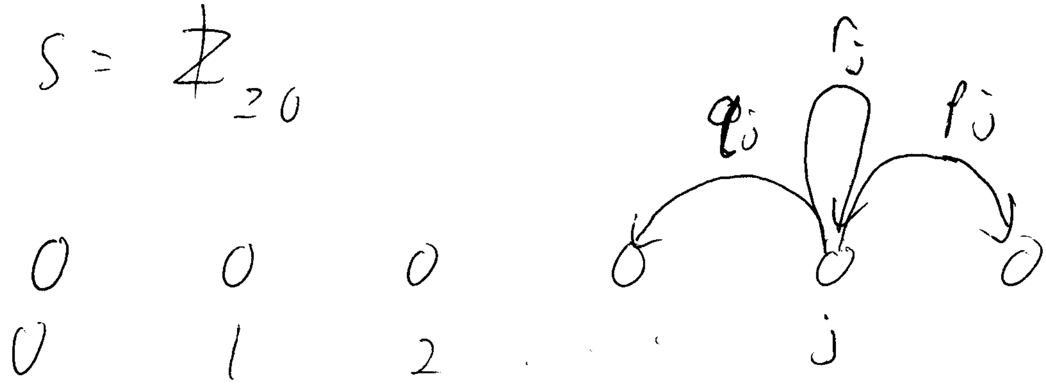
Example 2) Birth-death chain

$$\text{Prob}(X_{n+1} = j+1 \mid X_n = j) = p_j$$

$$\text{Prob}(X_{n+1} = j \mid X_n = j) = r_j \qquad \text{for } j \geq 1$$

$$\text{Prob}(X_{n+1} = j-1 \mid X_n = j) = q_j \qquad p_j + q_j + r_j = 1$$

$S = \mathbb{Z}_{\geq 0}$



Special case for $j = 0$

$$\text{Prob}(X_{n+1} = 1 \mid X_n = 0) = p_0$$

$$\text{Prob}(X_{n+1} = 0 \mid X_n = 0) = r_0 \qquad p_0 + r_0 = 1$$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & & \end{array} \\ \begin{pmatrix} r_0 & p_0 & 0 & & & \\ q_1 & r_1 & p_1 & 0 & & 0 \\ 0 & q_2 & r_2 & p_2 & 0 & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & \ddots & \ddots \end{pmatrix} \end{array}$$

This MC is irreducible provided

1) $p_0 > 0$ (otherwise 0 is absorbing state)

2) $p_j > 0$ and $q_j > 0$ for all $j \geq 1$

If these conditions are violated, absorbing states are recurrent

... various special cases which can be decided by topology of connections.

Irreducible case:

A) Look for invariant measure $\vec{r}$

$$\vec{r} P = \vec{r}$$

$$r_j \geq 0$$

$$r_0 r_0 + r_1 q_1 = r_0$$

$$r_0 p_0 + r_1 r_1 + r_2 q_2 = r_1$$

$$\vdots$$

$$r_{j-1} p_{j-1} + r_j r_j + r_{j+1} q_{j+1} = r_j$$

$$\vdots$$

Natural to try ~~recur~~ to obtain solution by solving equations recursively... huge mess.

Better idea: Try to look for a invariant measure which satisfies <u>detailed balance</u> (Haken, <u>Synergetics</u>)

- not guaranteed to work
- it's nice when it does!

General rule for invariant measure is "full balance"

$$\vec{r} P = \vec{r} \iff \sum_{i \in S} r_i P_{ij} = r_j \quad \text{for } j \in S$$

$$\iff \sum_{i \in S} r_i P_{ij} = \sum_{k \in S} r_j P_{jk} \quad \text{for } j \in S$$

$$\iff \sum_{\substack{i \in S \\ i \neq j}} r_i P_{ij} = \sum_{\substack{k \in S \\ k \neq j}} r_j P_{jk} \quad \text{for } j \in S$$

<u>Total</u> ~~out~~ flux ~~out of~~ into state j

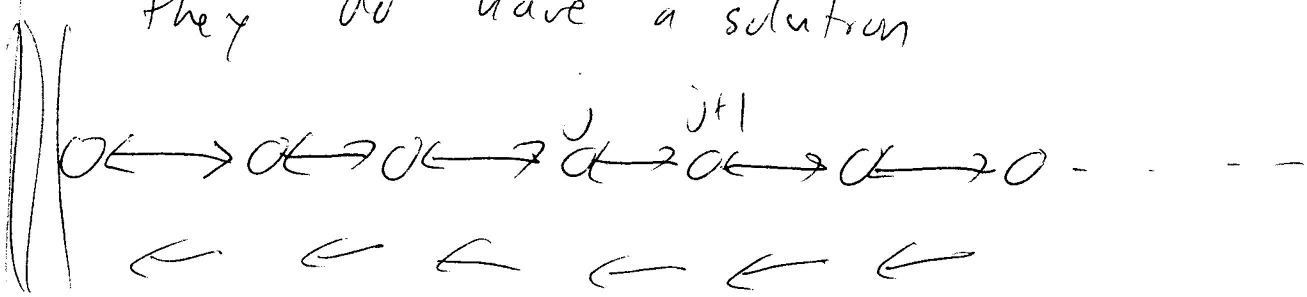
Total out flux ~~into~~ out of state j.

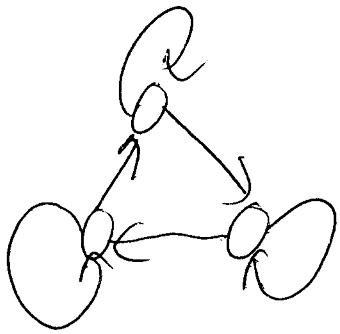
Detailed balance is special case where

$$r_i P_{ij} = r_j P_{ji} \quad \text{for all } i \neq j$$

$\uparrow$ flux $i \to j$ $\qquad$ $\updownarrow$ flux $j \to i$

These equations are <u>overdetermined</u>
- but for some MC structures they do have a solution





; not detailed balance

Don't have to worry about being clever.
Because invar measure is unique up to constant multiple for irreducible MC, you can just try to solve the detailed balance eqns: If you get self-consistent result, it's the right invariant measure.

Try detailed balance

$$r_j p_j = r_{j+1} q_{j+1} \quad \text{for } j \geq 0$$

$$\cancel{r_{j+1} = \frac{q_{j+1}}{p_j} r_j \quad \text{for } j \geq 0}$$

$$\cancel{r_j = \prod_{i=0}^{j-1} \frac{p_i}{q}}$$

$$r_{j+1} = \frac{p_j}{q_{j+1}} r_j \quad \text{for } j \geq 0$$

$$r_j = \left(\prod_{i=0}^{j-1} \frac{p_i}{q_{i+1}} \right) r_0 \quad \text{for } j \geq 1$$

Choose $r_0 = 1$

$$r_j = \prod_{i=0}^{j-1} \frac{p_i}{q_{i+1}}$$

This satisfies detailed balance eqs, so is the invariant measure for our irreducible MC!

Do I have stationary distribution?

Precisely iff the invar measure is normalizable

$$\vec{\pi} = \frac{\vec{r}}{\sum_{j \in S} r_j}$$

$$\sum_{j \in S} r_j < \infty \iff \sum_{j=0}^{\infty} r_j < \infty$$

$$\iff \sum_{j=0}^{\infty} \left(\prod_{i=0}^{j-1} \frac{p_i}{q_{i+1}} \right) < \infty$$

$\iff$ existence of stat. dist. $\vec{\pi}$

$\iff$ positive recurrence

B) Test for transience of MC

Choose $i=0$ as the reference state

let Q be the matrix obtained by deleting row 0 and column 0 from P

Look for solutions to $Q\vec{x} = \vec{x}$

$\vec{x} = \{x_j\}_{j=1}^{\infty}$

The tridiagonal system that results is not hard to solve by recursion

- Same as our calculation for absorption in the finite state birth-death chain.

Result: $x_j = x_1 \underbrace{\sum_{k=1}^{j} \gamma_k}$

where $\gamma_k = \prod_{1 \le k' \le k-1} \frac{q_{k'}}{p_{k'}}$

What I want to know is if there is a solution $\vec{x}$ with all $0 \le x_j \le 1$ for all $j \ge 1$

A nonzero solution $\vec{0} \le \vec{x} \le \vec{1}$ exists iff

$$\sum_{k=1}^{\infty} \gamma_k < \infty \iff \text{transience.}$$

To summarize results, it is helpful to note that $\gamma_k = p_0 r_{k-1}^{-1} p_{k-1}^{-1} = \dfrac{p_0}{r_{k-1} p_{k-1}}$

Summary for birth-death chain on irreducible case

1) Positive recurrent $\iff \displaystyle\sum_{j=0}^{\infty} r_j < \infty$

2) Transient $\iff \displaystyle\sum_{j=0}^{\infty} \frac{1}{r_j p_j} < \infty$

3) Null recurrent $\iff \displaystyle\sum_{j=0}^{\infty} r_j = \infty$ and $\displaystyle\sum_{j=0}^{\infty} \frac{1}{r_j p_j} = \infty$

where $r_j = \displaystyle\prod_{i=0}^{j-1} \frac{p_i}{q_{i+1}}$

Branching processes Lawler Sec. 2.4

X_n = # agents at time n

Each agent gives rise to a certain random number of agents at next time step. The offspring of each agent are iid rv's.

General context: X_n = population at generation n

Also for physical processes like nuclear fission, queueing.

each customer begets new customers during the time he is being served.

Note one can allow parent to survive to next generation – just consider it as own offspring.

Age structure neglected!

Stochastic update rule:

$$X_{n+1} = \sum_{k=1}^{X_n} Y_{n,k} \qquad X_0 = 1$$

where $\{Y_{n,k}\}$ are iid rv representing # offspring of an agent in next generation (including parent if it survives)

This is a countable-state Markov chain

Because we're dealing with random sums with nice recursive structure, generating functions work well.

$$P_{X,n}(s) = \mathbb{E}\, s^{X_n} \equiv \langle s^{X_n} \rangle$$

$P_Y(s) = \mathbb{E}\, s^Y$: gen fn for # offspring of one parent.

$$P_{X,n+1}(s) = \mathbb{E}\, s^{X_{n+1}} = \mathbb{E}\, s^{\sum_{k=1}^{X_n} Y_{n,k}}$$

$$= P_{X,n}(P_Y(s))$$

This follows from the fact that

if $Z = \sum_{k=1}^{N} Y_j$ with N, Y_j ~~a~~ rvs

$\{Y_j\}$ are iid

then $P_Z(s) = P_N(P_Y(s))$

Proceed by induction to get:

$$\boxed{P_{X,n}(s) = P_Y^{\circ n}(s)} \quad \left(\begin{array}{c}\text{compose } P_Y \ n \\ \text{times}\end{array}\right)$$

assuming we start with $X_0 = 1$.

$P_{X,1}(s) = P_Y(s)$

$P_{X,2}(s) = P_Y(P_Y(s))$

$P_{X,3}(s) = P_Y(P_Y(P_Y(s)))$ . . . .

Homework 3 will be posted
7 PM tonight
due April 2