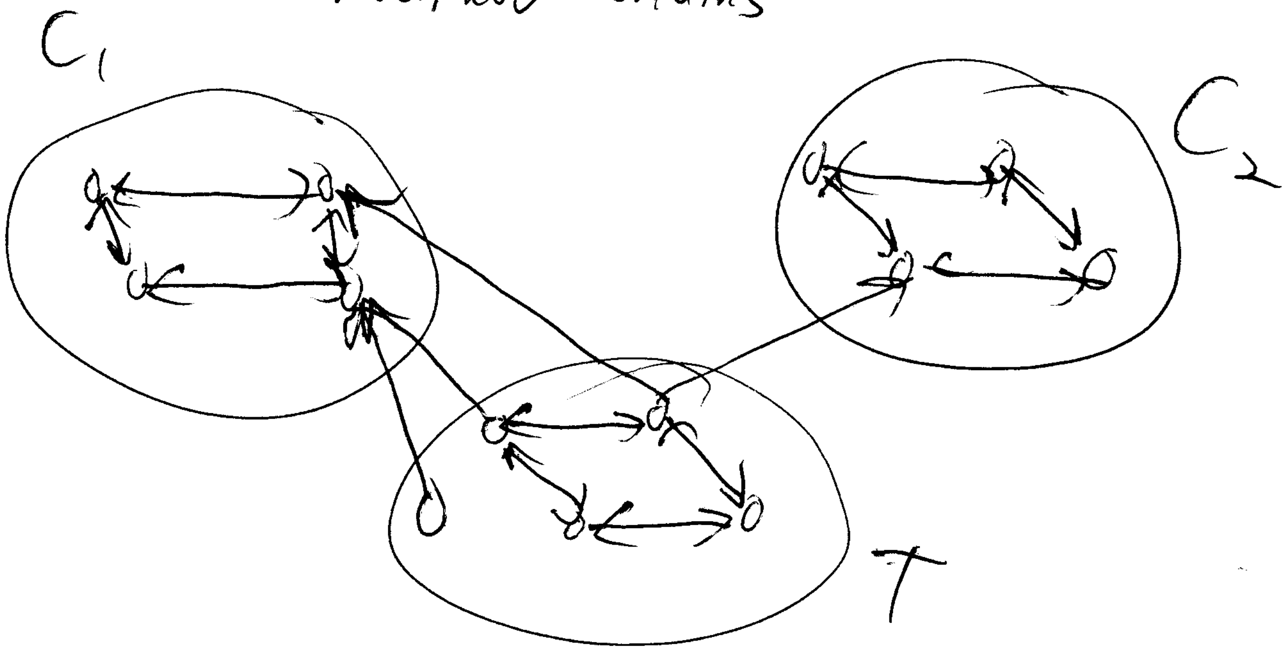
03/04/04 Cost/reward sums in transient states of Finite State, Discrete Time Markov Chains



Functions of MC. $f: S \to \mathbb{R}$

$\sum_n f(X_j)$ behave like?

Long run: use limit distributions of the recurrent classes you can get absorbed into

Also of interest:

$$\sum_{n=0}^{\tau-1} f(X_n)$$ where τ is the first time that the set of transient states is left,

$\tau = \min\{n \geq 0 : X_n \notin T\}$

Particular interest:

a) $f \equiv 1 \Rightarrow$ the sum = # steps until hit a recurrent class

b) $f(j) = \delta_{ij} \Rightarrow$ the sum = # epochs spent in state i before hit a recurrent class

We'll focus on expected values:

$$W_i = \mathbb{E}\left(\sum_{n=0}^{T-1} f(X_n) \,\middle|\, X_0 = i\right)$$

First-step analysis

$$W_i = f(i) + \mathbb{E}\left(\sum_{n=1}^{T-1} f(X_n) \,\middle|\, X_0 = i\right)$$

$$= f(i) + \sum_{j\in S} \mathbb{E}\left(\sum_{n=1}^{T-1} f(X_n) \,\middle|\, X_1 = j, X_0 = i\right) \mathrm{Prob}(X_1 = j \mid X_0 = i)$$

(used: $\mathbb{E}(Y \mid A) = \sum_{j\in S} \mathbb{E}(Y \mid A, Z = j) \times \mathrm{Prob}(Z = j \mid A)$)

Related to $\mathbb{E} f(X) = \sum_{j\in S} f(j)\,\mathrm{Prob}(X = j)$

$$w_i = f(i) + \sum_{j \in S} \mathbb{E}\left(\sum_{n=1}^{T-1} f(X_n) \,\Big|\, X_1 = j \right) \text{Prob}(X_1 = j \mid X_0 = i)$$

By Markov property

$$\cancel{\boxed{w_i = f(i) + \sum_{j \in S} w_j P_{ij}}}$$

~~In terms of matrices~~

$$\cancel{\vec{w} = \vec{f} + P\vec{w}}$$

~~Or~~ $\cancel{\vec{w}}$

Break into cases:

If $j \in T$, then $\mathbb{E}\left(\sum_{n=1}^{T-1} f(X_n) \,\Big|\, X_1 = j \right) = w_j$

If $j \notin T \Rightarrow T = 1 \Rightarrow \sum_{n=1}^{T-1} f(X_n) = 0$

$$\boxed{w_i = f(i) + \sum_{j \in T} w_j P_{ij}}$$

$$\vec{w} = \vec{f} + Q\vec{w}$$

where Q is submatrix of P corresponding to transient states.

$$\boxed{\vec{w} = (I - Q)^{-1} \vec{f}}$$

$$\vec{w} = \sum_{n=0}^{\infty} Q^n \vec{f}$$

Recall that $(Q^n)_{ij} = \text{Prob}(X_n = j \mid X_0 = i)$

$(Q^n \vec{f})_i$ = expected return at nth step, if start from i.

Special cases:

$$\left((I-Q)^{-1} \vec{1}\right)_i = \mathbb{E}(T \mid X_0 = i)$$

$\left((I-Q)^{-1}\right)_{ij}$ = expected amount of time spent in state j, given that $X_0 = i$

Remarks: 1) Calculate Variances of & $\sum_{n=0}^{T-1} f(X_n)$ using similar techniques

Bhat + Miller, Elements of Applied Stochastic Processes

Sec. 2.1 + 2.6.

Derivation flawed, result OK.

Apply it to credit management.

Remark 2): Use similar techniques to answer questions like:

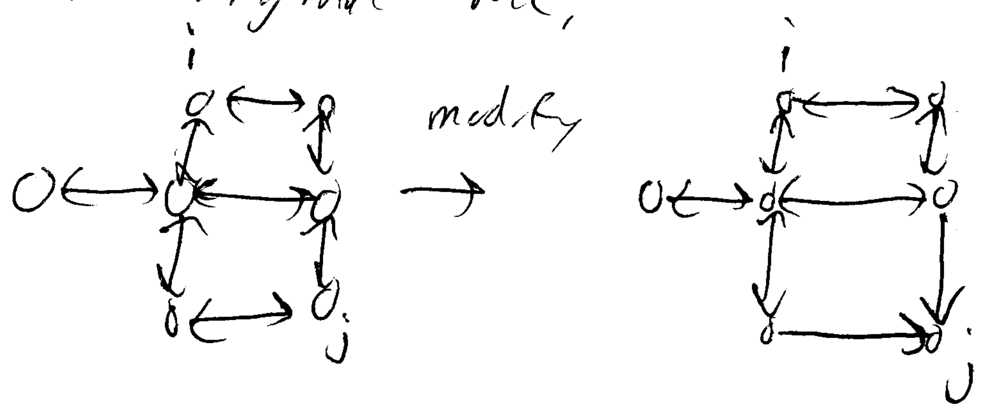
a) Starting from state i, what is the expected amount of time until I hit state j?

b) Starting from state i, how many times on avg do I visit state k before visiting j?

Can do this even if i, j are arbitrary elements of MC.

- consider a modified MC where j is made an absorbing state.

Then apply formulas to modified MC where is now transient if $i \to j$ in original MC.



This concludes finite state MC

All basic questions stated.

Now enrich stochastic models

Part 2 — infinite states (but discrete, countable)
— continuous time

Part 3 — continuous state space

Countable State, Discrete Time MCs

- infinitely many states; $S = \mathbb{Z}_{\geq 0}$, or $S = \mathbb{Z}$
  but discrete
- focus on time-homogenous case

References: Lawler Ch. 2

Karlin + Taylor Ch. 2 + 3

Examples: 1) Lifetime of products

2) Distribution of ~~d~~ clumped flaws in a product

3) Capital or asset models

4) Populations

5) Queues

6) Random walks on unbounded graphs

Lets re-examine concepts from finite state MC → what carries over and what needs modification?

Finite-time formulas (Chapman-Kolmogorov, etc.)

- work the same way
- probability transition matrix P is infinite-dimensional

$$\vec{\phi}_j^{(n)} = \text{Prob}(X_n = j)$$

$$\vec{\phi}^{(n)} = \vec{\phi}^{(0)} P^n$$

$$\phi_j^{(n)} = \sum_{k \in S} \phi_k^{(0)} (P^n)_{kj}$$

$$(P^2)_{ij} = \sum_{k \in S} P_{ik} P_{kj} \quad \text{etc.}$$

Only worry.. do infinite sums over S converge?

Yes because row sums are 1, and probabilities all bounded by 1.

Long-time properties? Transience + recurrence.

One new possibility for ∞-state MC is to have an irreducible MC which is transient.

Recurrence: 2 types

first return to state j.

Positive recurrence: $\mathbb{E}[T_j(1) \mid X_0 = j] < \infty$

- expected time to return $< \infty$

Null recurrence: $\mathbb{E}[T_j(1) \mid X_0 = j] = \infty$

but $\text{Prob}(T_j(1) < \infty) = 1$.

- slowly decaying PDF for $T_j(1)$

Recall transience of state $j \iff$

$\text{Prob}(T_j(1) < \infty) < 1$.

1) What happens in MC of these 3 types?

2) How do I tell what kind of MC I have?

Two mathematical approaches to these ?'s

First approach is useful if you can get explicit formulas for p^n for arbitrary n
- random walks (Karlin + Taylor Sec. 2.6)
- Karlin + Taylor Sec. 2.5
  Resnick Sec. 2.6

Develop a recursion formula linking
$\text{Prob}(X_n = j \mid X_0 = i)$ to

$\text{Prob}(X_n = j$ for the first time $\mid X_0 = i)$
at time n

First-passage probability

Develop probability gen fns.... get some results:

A) State i is transient $\Longleftrightarrow \sum_{n=0}^{\infty} (p^n)_{ii} < \infty$

B) State i is recurrent $\Longleftrightarrow \sum_{n=0}^{\infty} (p^n)_{ii} = \infty$

C) If T_j = # epochs spent in state j

Then if $j \in C_k$ (a recurrent class)

~~and~~ then $\text{Prob}(T_j = \infty \mid X_0 = i) = 1$
if $i \in C_k$

If $j \in T$ (transient)

then $\mathbb{E}(T_j \mid X_0 = i) < \infty$ for any $i \in S$

If $X_0 = j$, then T_j obeys a geometric dist.

If $X_0 \neq j$, then T_j obeys a similar dist.

Other approach does not require us to know P^n.

First we'll look at what happens in each type of MC, then we'll figure out how to tell what kind of MC a given model is

1) A) Suppose we have an irreducible MC which has ≥ 1 positive recurrent state j.

Then the whole MC is positive recurrent and has a unique stationary distribution $\vec{\pi}$

$$\pi_j > 0, \quad \sum_{j \in S} \pi_j = 1, \quad \vec{\pi} \cdot P = \vec{\pi}$$

Also, $\vec{\pi}$ acts as a limit distribution

$$\lim_{n \to \infty} (P^n)_{ij} = \pi_j$$

~~Proof~~ $$\lim_{n \to \infty} \vec{\phi} \cdot P^n = \vec{\pi}$$

Proof? Our proof for finite state MC works, built on the hypothesis $\boxed{\pi_j = \frac{1}{\mathbb{E}(T_j(1) \mid X_0 = j)}}$

which is proven to be true.

That proved existence, uniqueness of $\vec{\pi}$.

Limit? Can't use Perron-Frobenius for ∞-dim matrix

Coupling argument works.

B) Recurrent (but not nec. pos. recurrent)

- then what survives from finite time proof is existence of invariant measure

$\vec{r}$: $\quad r_j \geq 0$

$$\vec{r} \cdot P = \vec{r}$$

Invariant measure is unique up to constant multiple:

If $\vec{\mu}, \vec{r}$ are invar meas, then $\vec{\mu} = c\,\vec{r}$ for some $c \geq 0$.

Positive recurrent:

$$\lim_{n\to\infty} (P^n)_{ij} = \pi_j \qquad \lim_{n\to\infty} \vec{\phi} \cdot P^n = \vec{\pi}$$

Null recurrent:

$$\lim_{n\to\infty} (P^n)_{ij} = 0$$