03/01/04

What happens in the long run if I start in a state $i \in T$.

(Lawler Sec. 1.3)

Sometimes helpful (not mandatory) to write the transition matrix for a reducible MC in canonical form,

$$\begin{array}{ccc} & T^c & T \\ \text{recurrent } T^c & \left[\begin{array}{c} \bar{P} \\ S \end{array}\right. & \left.\begin{array}{c} 0 \\ Q \end{array}\right] = P \\ \text{transient } T & & \end{array}$$

Arrange recurrent states to group them by communication classes so then $\bar{P}$ will be block diagonal

$$\bar{P} \quad \begin{array}{c|cccc} & C_1 & C_2 & & C_M \\ \hline C_1 & \bar{P}^{(1)} & 0 & 0 & \\ C_2 & 0 & \bar{P}^{(2)} & 0 & \\ \vdots & 0 & 0 & \ddots & \\ C_M & & & & \bar{P}^{(M)} \end{array}$$

Question: If I start at $i \in T$, then what is the first state $j \in T^c$ that I reach?

This is important because it tells us what recurrent class we end up in, which in turn describes the long term properties.

Let $U_{ij} = \text{Prob}(X_{\mathcal{I}} = j \mid X_0 = i)$

for $i \in T$, $j \in T^c$.

Where $\mathcal{I}$ is the time at which the MC first leave transient states,

$\mathcal{I} = \inf\{n \in \mathbb{Z}_{\geq 0} : X_n \notin T\}$

To calculate U_{ij}, we will use

<u>first-step analysis</u>

- study + break down the MC dynamics based on what happens at first step.

$$U_{ij} = \mathrm{Prob}(X_\tau = j \mid X_0 = i)$$

$$= \sum_{k \in S} \mathrm{Prob}(X_\tau = j \text{ and } X_1 = k \mid X_0 = i)$$

$$\mathrm{Prob}(A \text{ and } B \mid C) = \mathrm{Prob}(A \mid B \text{ and } C) \times \mathrm{Prob}(B \mid C)$$

$$= \sum_{k \in S} \mathrm{Prob}(X_\tau = j \mid X_1 = k \text{ and } X_0 = i)\, \mathrm{Prob}(X_1 = k \mid X_0 = i)$$

$$= \sum_{k \in S} \underline{\mathrm{Prob}(X_\tau = j \mid X_1 = k)}\, \underbrace{\mathrm{Prob}(X_1 = k \mid X_0 = i)}_{= P_{ik}}$$

What do these factors look like for various k?

Suppose $k \in T^c$ and $k \neq j \Rightarrow \tau = 1$

$$\Rightarrow \mathrm{Prob}(X_\tau = j \mid X_1 = k) = 0$$

Suppose $k = j \Rightarrow \tau = 1$ (since $j \in T^c$)

$$\mathrm{Prob}(X_\tau = j \mid X_1 = k) = 1$$

Suppose $k \in T$, then $\mathrm{Prob}(X_\tau = j \mid X_1 = k) = U_{kj}$

Combining these facts

$$\boxed{U_{ij} = P_{ij} + \sum_{k\in T} P_{ik} U_{kj}} \quad \text{for } i\in T, \; j\in T^c$$

This can be also be written

$$U_{ij} = S_{ij} + \sum_{k\in T} Q_{ik} U_{kj}$$

where Q, S are the submatrices of P,

$U = S + QV$ as matrices,

$(I-Q)\,V = S$

$$\boxed{U = (I-Q)^{-1} S}$$

But is $I-Q$ really invertible? Yes,

$$(I-Q)^{-1} = \sum_{n=0}^{\infty} Q^n \quad \text{if RHS exists,}$$

"Resolvent"

$(Q^n)_{ij} = \mathrm{Prob}(X_n = j \mid X_0 = i)$ decreases geometrically with n, This implies in particular $Q^n \to 0$ as $n \to \infty$ and so all evs λ of Q must satisfy $|\lambda| < 1$,

$\Rightarrow (I-Q)^{-1}$ exists

The formula $U = (I-Q)^{-1}S$ has an intuitive meaning

$$U = (I-Q)^{-1}S = \sum_{n=0}^{\infty} Q^n S$$

$$\cancel{= \sum_{n=0}^{\infty} \mathrm{Prob}(X_I = j \text{ and } I = }$$

$$U_{ij} = \sum_{n=0}^{\infty} (Q^n S)_{ij}$$

$$(Q^n S)_{ij} = \mathrm{Prob}(X_I = j \text{ and } I = n+1 \mid X_0 = i)$$

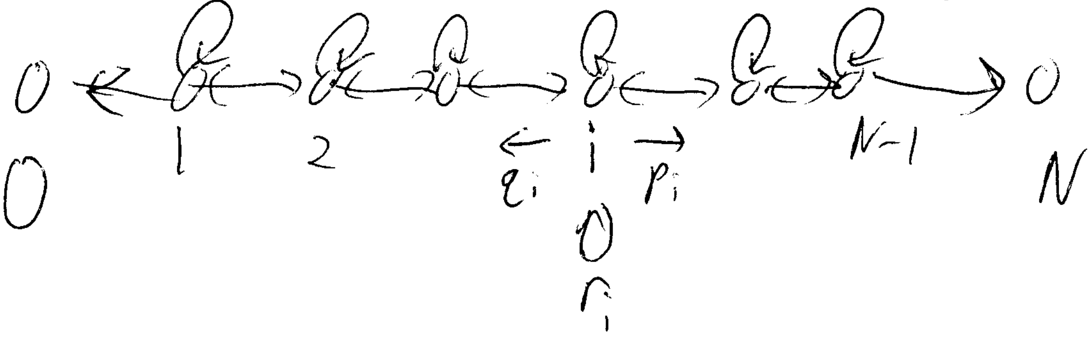
Birth-death chains

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ \\ N \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & \cdots & & & & N \end{array} \\ \left[\begin{array}{cccccccc} 1 & 0 & 0 & \cdots & & & & 0 \\ q_1 & r_1 & p_1 & 0 & \cdots & & & 0 \\ 0 & q_2 & r_2 & p_2 & 0 & \cdots & & 0 \\ & & & \ddots & & & & \\ & & & & & & & 0 \\ & & & & & q_{N-1} & r_{N-1} & p_{N-1} \\ & & & & & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$q_i + r_i + p_i = 1$$

q_i = prob for left move from i
p_i = prob for right move from i
r_i = prob for no move from i

The states 0 and N : absorbing



Starting in state i, whats the prob I eventually wind up in state 0 (or in state N)

Canonical form (not necessary)

$T = \{1, \dots, N-1\}$

$C_1 = \{0\}$

$C_2 = \{N\}$

$\bar{P}$

$$
\begin{array}{cc|c}
 & C_1 \quad C_2 & T \\
 & \begin{array}{cc} 0 & N \end{array} & \begin{array}{ccccc} 1 & 2 & \cdots & & N-1 \end{array} \\
\hline
\begin{array}{cc} C_1 & 0 \\ C_2 & N \end{array} & \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} & 0 \\
\hline
\begin{array}{c} 1 \\ 2 \\ \vdots \\ T \\ \vdots \\ \\ N-1 \end{array} & \underbrace{\begin{array}{cc} q_1 & 0 \\ 0 & 0 \\ & \\ 0 & \\ & \\ & \\ 0 & p_{N-1} \end{array}}_{S} & \underbrace{\begin{array}{ccccccc} r_1 & p_1 & 0 & \cdots & & & 0 \\ q_2 & r_2 & p_2 & 0 & \cdots & & 0 \\ 0 & & \ddots & & & & \\ & & & \ddots & & & \\ & & & 0 & q_{N-2} & r_{N-2} & p_{N-2} \\ 0 & \cdots & & & 0 & q_{N-1} & r_{N-1} \end{array}}_{Q}
\end{array}
$$

$U = (I - Q)^{-1} S$ ... hard to explicit formulas

$U_{ij} = \text{Prob}(X_T = j \mid X_0 = i)$ where

T is first time at which ~~the~~ $X_n \notin \{1, \ldots, N-1\}$

$$U_{ij} = P_{ij} + \sum_{k=1}^{N-1} P_{ik} U_{kj} \qquad \begin{array}{l} 1 \le i \le N-1 \\ j = 0 \text{ or } j = N. \end{array}$$

Solve by hand.

For $2 \le i \le N-2$

$$U_{ij} = q_i U_{i-1,j} + r_i U_{ij} + p_i U_{i+1,j}$$

Intuitive meaning!

Can also use this eqn for all $1 \le i \le N-1$ if I set (define)

$$U_{00} = 1, \quad U_{0N} = 0, \quad U_{N0} = 0, \quad U_{NN} = 1$$

This is now a system of linear difference eqns w/ BCs.

- techniques: bare hands
  - ~~gf~~ generating functions (esp. if recursive)
  - Bender Orszag Ch. 2 or 3

Because $r_i = 1 - p_i - q_i$

$$q_i (U_{i-1,j} - U_{ij}) + p_i (U_{i+1,j} - U_{ij}) = 0$$

Define $V_{ij} = U_{ij} - U_{i-1,j}$ for $1 \le i \le N$

$$- q_i V_{ij} + p_i V_{i+1,j} = 0$$

$$V_{i+1,j} = \frac{q_i}{p_i} V_{ij}$$

$$V_{i,j} = \left(\prod_{1 \le k \le i-1} \frac{q_k}{p_k} \right) V_{1,j}$$

$$\gamma_i = \prod_{1 \le k \le i-1} \frac{q_k}{p_k}$$

Get $V_{1,j}$ from using BC's

$$\sum_{i=1}^{N} V_{ij} = U_{Nj} - U_{0j} = \delta_{Nj} - \delta_{0j}$$

$$\left(\sum_{k=1}^{N} \gamma_k \right) V_{1j} = \delta_{Nj} - \delta_{0j}$$

$$V_{1j} = \frac{\delta_{Nj} - \delta_{0j}}{\sum_{k=1}^{N} \gamma_k}$$

$$V_{ij} = \gamma_i \frac{(\delta_{Nj} - \delta_{0j})}{\sum_{k=1}^{N} \gamma_k}$$

$$U_{ij} = \sum_{k=1}^{i} V_{kj} + U_{0j}$$

$$U_{i0} = Prob(X_T = 0 \mid X_0 = i) = \frac{\sum_{k'=i+1}^{N} \gamma_{k'}}{\sum_{k=1}^{N} \gamma_k}$$

$$U_{iN} = Prob(X_T = N \mid X_0 = i) = \frac{\sum_{k'=1}^{i} \gamma_{k'}}{\sum_{k=1}^{N} \gamma_k}$$

where $\gamma_i = \prod_{1 \le k \le i-1} \frac{q_k}{p_k}$

These techniques can also be used to answer

$$Prob(X_T \in C_k \mid X_0 = i) \quad \text{for } i \in T$$

C_k is a recurrent class

Could add up probabilities for each state in C_k or...

write a condensed MC where C_k is collapsed into 1 state.

Another observation ... later

Another question of interest:

Suppose there is a function f associated to MC.

How does $\sum_{n=1}^{N} f(X_n)$ behave ~~for~~?

Examples: income
deaths
productivity,

For large N, $\frac{1}{N} \sum_{n=1}^{N} f(X_n)$ behaves like?

→ first figure out the probabilities that you wind up in the various recurrent classes

→ given you end up in a recurrent class C_k,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(X_n) = \sum_{j\in S} \pi_j^{(k)} f(j) \quad \text{w/ prob } 1$$

stat dist. assoc. to class C_k.

(Karlin + Taylor Ch. 3)

HW 2 can be turned in on

Wed March 17.