02/26/04 This MC is irreducible, aperiodic, finite state

$\therefore \lim_{n \to \infty} Prob(X_n = j) = \pi_j$

Where $\vec{\pi}$ is the unique stat dist.

Let's calculate $\vec{\pi}$ then use for questions.

$\vec{\pi} \cdot P = \vec{\pi}$ with $\sum_{j=0}^{M} \pi_j = 1$

~~$\pi_j P$~~

0th eqn: $p \sum_{j=0}^{M} \pi_j = \pi_0$

$1 \le j' \le M-1$: $(1-p)\,\pi_{j'-1} = \pi_{j'}$

Mth eqn: $(1-p)(\pi_{M-1} + \pi_M) = \pi_M$

Middle eqns: $\pi_j = (1-p)^j \pi_0$ for $1 \le j \le M-1$

0th eqn: $\pi_0 = p$

$\pi_j = p(1-p)^j$

$\pi_M = 1 - \sum_{j=0}^{M-1} \pi_j = 1 - p \sum_{j=0}^{M-1} (1-p)^j$

$= 1 - p\left(\frac{1-(1-p)^M}{p}\right) = (1-p)^M$

This describes $\vec{\pi}$.

Consider the question of how many products are inspected first.

I = fraction of items inspected

$$= \lim_{n\to\infty} \frac{\#\text{ inspections } (n)}{\#\text{ products that pass through in } n \text{ inspections.}}$$

T_n = # products passing the inspection station between inspection # n and inspection # n+1

$$T_n = f(X_n) = \begin{cases} 1 & \text{for } X_n \neq M \\ r & \text{for } X_n = M \end{cases}$$

$$I = \lim_{n\to\infty} \frac{n}{\sum_{j=1}^{n} T_j}$$

Probability theory: Law of Large Numbers

If $\{Y_j\}_{j=0}^{\infty}$ are iid, nice r. variables

then $\frac{1}{n}\sum_{j=0}^{n} Y_j \xrightarrow[n\to\infty]{} \langle Y\rangle$ almost surely (prob = 1)

Law of Large Numbers for MC
(Resnick 2.12)

If $\{Y_n\}_{n=0}^{\infty}$ is a FS HDT MC

which is ergodic (aperiodic + irreducible + (positive recurrent))

then for deterministic functions f

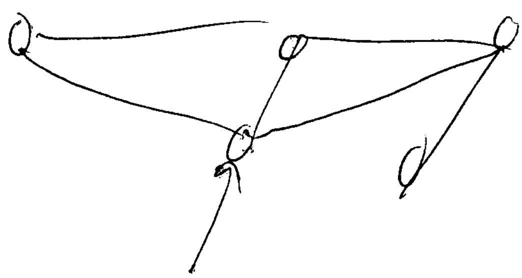
$$\frac{1}{N}\sum_{n\geq 1}^{N} f(Y_n) \to \sum_{j\in S} \pi_j^{(Y)} f(j)$$

almost surely (w/ prob. 1)

where $\vec{\pi}$ is stat. dist.

Idea of proof: Reduce to LLN for iid rv's.

- dissection principle (Resnick):
  The trajectories of MC between successive visits to a reference state are independent of each other



Apply the LLN for MC

$$I = \lim_{n\to\infty} \frac{n}{\sum_{j=1}^{n} T_j} = \frac{1}{\lim_{n\to\infty} \frac{\sum_{j=1}^{n} T_j}{n}}$$

$$= \frac{1}{\sum_{j=0}^{M} \pi_j f(j)}$$

$$f(j) = \begin{cases} 1 & \text{for } 0 \le j \le M-1 \\ r & \text{for } j = M \end{cases}$$

$$= \frac{1}{\left(\sum_{j=0}^{M-1} \pi_j\right) 1 + \pi_M r} = \frac{1}{1 - \pi_M + \pi_M r}$$

$$= \frac{1}{1 + (r-1)\pi_M}$$

$$I = \frac{1}{1 + (r-1)(1-p)^M} = \text{fraction of products inspected}$$

How about fraction of shipped products that are defective?

$$D = \frac{\text{\# products shipped + defective}}{\text{\# products shipped}}$$

$$= \frac{\sum_{n=1}^{N} D_n}{\sum_{n=1}^{N} S_n}$$

D_n = # defective products shipped + defective right from ~~from~~ after nth inspection up to (but not including) (n+1)st inspection

$D_n = 0$ when $0 \le X_n \le M-1$

D_n is random when $X_n = M$!

Can't write D_n as a deterministic function of X_n.

$$\text{Prob}(D_n = d) = \begin{cases} \delta_{d0} & \text{when } X_n \ne M \\ b(d; r-1, p) & \text{when } X_n = M \end{cases}$$

binomial dist.

$$\delta_{ij} = 1 \quad \text{for } i=j$$
$$= 0 \quad \text{for } i \ne j$$

S_n = # products shipped from nth inspection up to, but not including inspection # n+1

$S_n = 0$ if $X_n = 0$

$S_n = 1$ if $1 \le X_n \le M-1$

$S_n = r$ if $X_n = M$

$$S_n = g(X_n)$$

$$g(j) = \begin{cases} 0 & \text{for } j=0 \\ 1 & \text{for } 1 \le j \le M-1 \\ r & \text{for } j = M \end{cases}$$

$$D = \lim_{N\to\infty} \frac{\sum_{n=1}^{N} D_n}{\sum_{n=1}^{N} S_n} = \lim_{N\to\infty} \frac{\frac{1}{N}\sum_{n=1}^{N} D_n}{\frac{1}{N}\sum_{n=1}^{N} S_n}$$

LLN for MC:

$$\frac{1}{N}\sum_{n=1}^{N} S_n \xrightarrow[N\to\infty]{} \sum_{j=0}^{M} \pi_j \, g(j) \quad \text{w/ prob. 1}$$

By viewing the extended MC $\{X_n, D_n\}$ and apply LLN

$$\frac{1}{N}\sum_{n=1}^{N} D_n \xrightarrow[N\to\infty]{} \sum_{j=0}^{M} \pi_j \langle D_n \mid X_n = j\rangle$$

$$\langle D_n \mid X_n = j\rangle = 0 \quad \text{for } 0 \le j \le M-1$$

$$\langle D_n \mid X_n = M\rangle = (r-1)p$$

$$D = \frac{\sum_{j=0}^{M} \pi_j \langle D_n \mid X_n = j\rangle}{\sum_{j=0}^{M} \pi_j \, g(j)}$$

$$= \frac{\left(\sum_{j=0}^{M-1} \pi_j\right) 0 + \pi_M (r-1)p}{\pi_0 (0) + \left(\sum_{j=1}^{M-1} \pi_j\right) 1 + \pi_M r}$$

$$= \frac{\pi_M (r-1)p}{1 - \pi_M - \pi_0 + r\pi_M} = \frac{\pi_M (r-1)p}{1 + (r-1)\pi_M - p}$$

$$= \frac{(1-p)^M (r-1)p}{1-p+(r-1)(1-p)^M} = \frac{(r-1)p(1-p)^{M+1}}{1+(r-1)(1-p)^{M-1}}$$

How is D related to I?

$$(r-1)(1-p)^M = \frac{1-I}{I}$$ by inverting the eqn for I.

Plugging into D

$$D = \frac{p\,\frac{1-I}{I}}{1-p+\frac{1-I}{I}} = \frac{p-pI}{1-pI}$$

Independent of M + r!

Relates outgoing quality to effort.

How does this compare w/ ~~same~~ simple random sampling where each product is inspected w/ prob. I?

Fraction of products shipped

$=$ Prob(product is inspected + good, or not inspected)

$$= I(1-p) + 1 - I = 1 - Ip$$

Fraction of products that are defective + shipped

$=$ Prob(product not inspected and bad)

$$= (1-I)\,p$$

For simple random sampling,

$$D = \frac{(1-I)p}{1-Ip} = \frac{p-pI}{1-Ip}$$

Same as for regular adaptive sampling scheme!

No benefit because products defects independent.

- may do better for clumped defects → Homework!

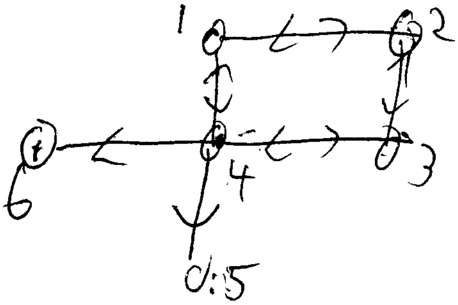
What about FS # DT MC which are not irreducible?

Trivial: Disconnected classes

; break into separate MC's.

More interesting is MC w/ 1-way paths between communication classes.

Random walk w/ absorbing BC



$\{1,2,3,4\}$: Transient
$\{5\}$: Recurrent
$\{6\}$: Recurrent

For reducible MC's: canonical decomposition

Classification of states:

A state $j \in S$ is <u>recurrent</u> if the MC starting from state j has prob $= 1$ to return to state j.

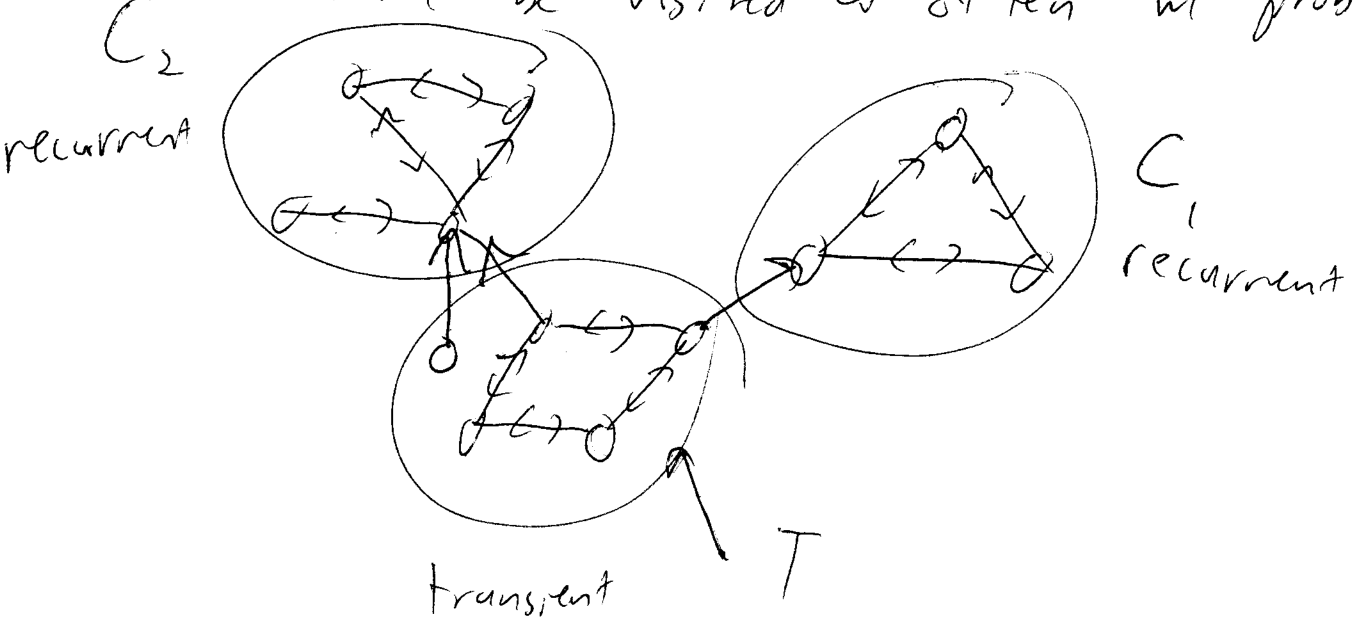
A state $j \in S$ is <u>transient</u> if the MC starting from state j has prob < 1 to return to state j.

Facts: 1) Recurrence + transience are class properties (same for all states in communication class)

~~2) In a recurrent class C_k, then sta~~

2) Recurrent state will be visited ∞ often (w/ prob. 1) if MC starts in that state.

3) In a recurrent class C_k, then starting from any state in that class, any other state will be visited ∞ often w/ prob. 1.



Decompose the MC into

$$S = T \cup_k C_k$$

all transient states (even if not 1 class)

Each C_k is a recurrent class,

If you start in a recurrent class, long-time properties can be calculated by just treating the recurrent class as its own irreducible MC