02/17/04 Stationary distributions for Finite State Discrete Time Homogenous Markov Chains

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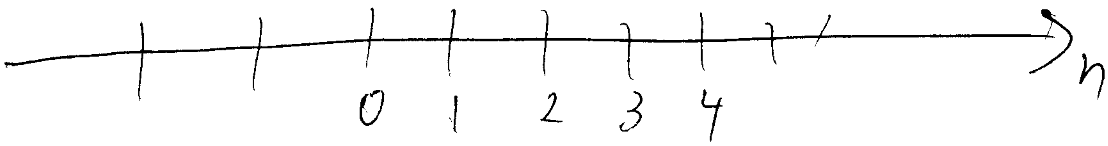
Numerical Simulation of FSDT Markov chains.

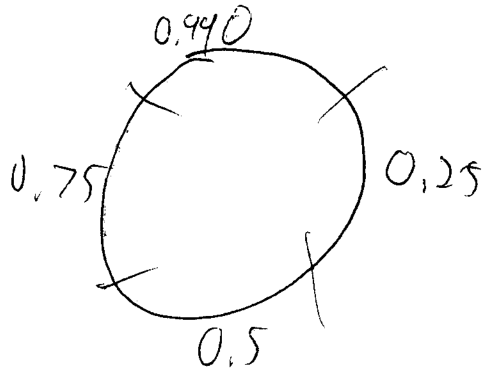


epoch: time step for DT MC

Start at epoch (time) $n = 0$:

$$\text{Prob}(X_0 = j) = \phi_j \quad \text{for } j \in S$$

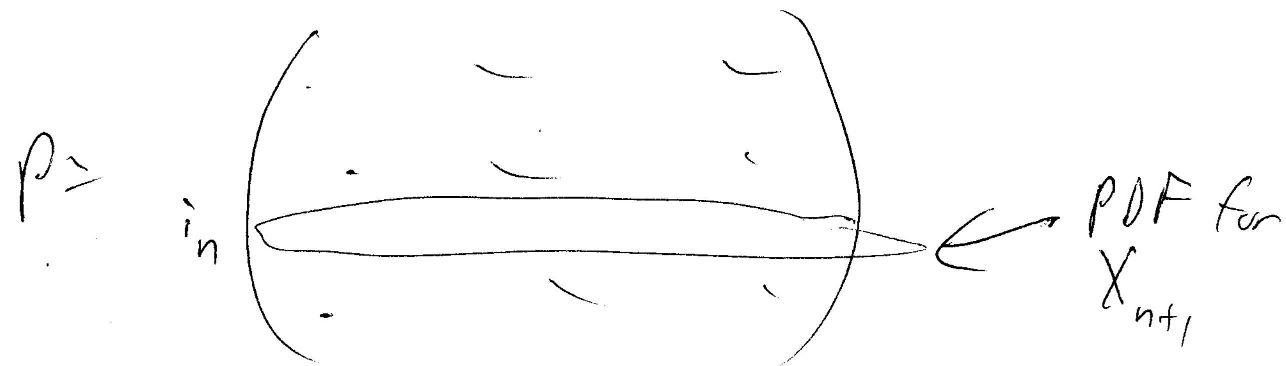
- Simulate as finite-valued rv



How simulate X_{n+1} once $X_0, X_1, \ldots, X_n$ are simulated?

$$\text{Prob}(X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n)$$
$$= \text{Prob}(X_{n+1} = j \mid X_n = i_n) = P_{i_n j}$$

So in other words, given the values $X_0 = i_0, \ldots, X_n = i_n$, simulate X_{n+1} as a finite-valued rv with (conditional) prob. dist. given by $P_{i_n j}$ for $j \in S$,



In a Markov chain model, how should we choose the initials prob. dist. $\vec{\emptyset}$ ($\emptyset_j = \text{Prob}(X_0 = j)$)

Sometimes you start the system in a certain way.

If start in state i_0, then choose $\emptyset_{i_0} = 1$
$\emptyset_i = 0$ for $i \neq i_0$

If however the epoch $n=0$ corresponds to a start of observations and there is no <u>control</u> over the initial data... then what?

Sometimes you can choose $\vec{\phi}$ so that the system is "statistically stationary" state.

- system is evolving, but its statistics are not.
- For a Markov chain: It is in statistically stationary state iff it's time-homogenous (so transition probability $P_{ij} = \text{Prob}(X_{n+1} = j \mid X_n = i)$ is independent of n)

  and $\text{Prob}(X_n = j)$ is independent of n

How can we choose $\vec{\phi}$ so that the system will be in a statistically stationary state?

Want $\phi_i = \text{Prob}(X_0 = i) = \text{Prob}(X_n = i)$ for all $n \geq 0$

$$\text{Prob}(X_n = i) = \cancel{\sum_{i' \in S} \phi_{i'} P_{i'i}}$$

$$= \sum_{i' \in S} \phi_{i'} (P^n)_{i'i}$$

So if we want $\phi_i = \text{Prob}(X_n = i)$:

$$\vec{\phi} = \vec{\phi} \cdot P^n \quad \text{for all } n \geq 0$$

This can be arranged if
$\vec{\phi}$ is a left eigenvector of P
with eigenvalue 1.

$$\vec{\phi} = \vec{\phi} \cdot P$$

Can we be sure such a $\vec{\phi}$ exists?

Recall that the row sums of P are $= 1$.

$$P \cdot \vec{1} = \vec{1} \qquad \vec{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \end{pmatrix}$$

Right eigenvector of eigenvalue 1
implies existence of left eigenvector of
eigenvalue 1.

(P not symmetric so left eigenvectors $\neq$
right eigenvectors)

But need $\phi_j \geq 0$ and $\sum_{j \in S} \phi_j = 1$.

Perron-Frobenius theory for positive matrices
(all entries <u>positive</u>)
- Karlin + Taylor <u>Appendix</u>, Lawler Ex. 1.15
- but doesn't work for matrices with some
zero entries

So we'll establish the existence/uniqueness of stationary distribution in another way.

To prepare, need to classify Markov chains.

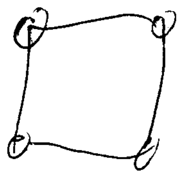
Two states $i, j \in S$ <u>communicate</u> if

$$(P^n)_{ij} > 0 \text{ and } (P^m)_{ji} > 0 \text{ for some } n, m \geq 0.$$

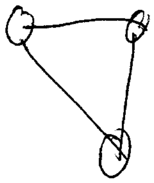
Communication is an <u>equivalence</u> relation so can divide S into <u>communication classes</u> such that all states in a communication class communicate with each other, but not with any other class.

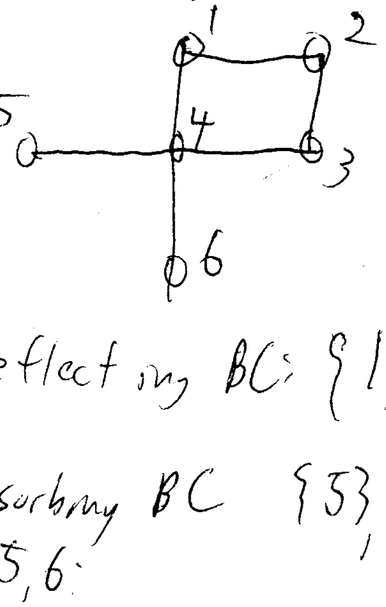
Example: Random walk on a graph

comm class



comm class





If reflecting BC: $\{1,2,3,4,5,6\}$

If absorbing BC at 5,6: $\{5\}, \{6\}, \{1,2,3,4\}$

If a Markov chain consists of a single communication class, it is said to be <u>irreducible</u>

(More classification later)

Proposition: Any finite state, discrete time, homogeneous, <u>irreducible</u> Markov chain has a unique stationary distribution $\vec{\pi}$.

Proof by construction:

A good hypothesis is that

$$\pi_j = \frac{1}{\mathbb{E}[T_j(1) \mid X_0 = j]}$$

where $T_j(1)$ is a random time at which state j is reached for first time after $n=0$.

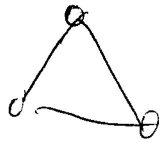
Fraction of time in state j

$$= \frac{1}{\text{avg amt of time between returns to } j}$$

Also, let $T_{j,k}$ be the random # epochs after $n=0$ spent in state k before hitting state j.

Also expect $\pi_k = \dfrac{\mathbb{E}[T_{j,k} \mid X_0 = j]}{\mathbb{E}[\tau_j(1) \mid X_0 = j]}$ *)

$\pi_2 = \frac{1}{2}$



$\pi_1 = 1/10$ $\quad$ $\pi_3 = \frac{2}{5}$

$$= \pi_j \, \mathbb{E}[T_{j,k} \mid X_0 = j]$$

$$T_{j,j} = 1$$

1 2 3 1 2 2 3 2 3 2 3 3 2 2 2 1

Will prove that the expression on RHS of *) is indeed ~~the~~ a stationary distribution.

Set $\tilde{\pi}_k = \dfrac{\mathbb{E}[T_{j,k} \mid X_0 = j]}{\mathbb{E}[\tau_j(1) \mid X_0 = j]}$ as my candidate for stationary dist (j is fixed)

Need to show: ✓ 1) $\tilde{\pi}_k \geq 0$ for $k \in S$

✓ 2) $\sum_{k \in S} \tilde{\pi}_k = 1$

3) $\vec{\tilde{\pi}} \cdot P = \vec{\tilde{\pi}}$

Then $\vec{\tilde{\pi}}$ would be a stationary dist.

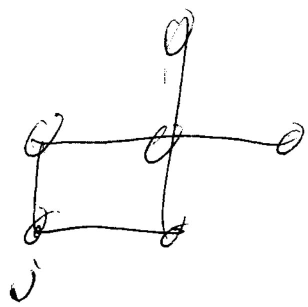
1) Obvious by defn

2) $\tau_j(1) = \sum_{k \in S} T_{j,k}$ $\qquad$ 1 2 3 2 3 3 3 2 1 $\quad$ $\tau_1(1) = 8$

So $\sum_{k \in S} \tilde{\pi}_k = \dfrac{\mathbb{E}[\sum_{k \in S} T_{j,k} \mid X_0 = j]}{\mathbb{E}[\tau_j(1) \mid X_0 = j]} = \dfrac{\mathbb{E}[\tau_j(1) \mid X_0 = j]}{\mathbb{E}[\tau_j(1) \mid X_0 = j]}$

We can conclude $\sum_{k\in S} \hat{\pi}_k = 1$ provided

$$\underset{\substack{\uparrow \\ \text{defn}}}{0 <} \mathbb{E}[T_j(1) \mid X_0 = j] < \infty$$



This can be shown by proving first that there exists $N \in \mathbb{Z}_{\geq 0}$ such that

$$\min_{i \in S} \max_{0 \leq n \leq N} (P^n)_{ij} > 0$$

(Lawler Ex 1.7)

This allows one to bound the probability to avoid state j over n time steps by a geometric distribution. (has finite mean)

So $T_j(1)$ also has finite mean.

This proves $\sum_{k\in S} \pi_k = 1$.

3) Enough to check that $\vec{r} \cdot P = \vec{r}$

for $r_k = \mathbb{E}[\, T_{j,k} \mid X_0 = j \,]$

because $\widehat{\widetilde{\pi}} = \dfrac{\vec{r}}{\sum_{k \in S} r_k}$

$\vec{r}$ will thereby be shown to be "invariant measure":

$$r_k \geq 0$$
$$\vec{r} \cdot P = \vec{r}$$

(but no normalization)

$$r_k = \mathbb{E}\Big[\sum_{n=0}^{T_j(1)-1} I\{X_n = k\} \,\Big|\, X_0 = j \Big]$$

indicator function $= 1$ if arg true, $= 0$ if arg false

$$= \sum_{n=0}^{\infty} \mathrm{Prob}(X_n = k,\ T_j(1) > n \mid X_0 = j)$$

$$\left(\text{used: } \mathbb{E}\, I\{X_n = k\} = 1 \cdot \mathrm{Prob}(X_n = k) + 0 \cdot \mathrm{Prob}(X_n \neq k) = \mathrm{Prob}(X_n = k) \right)$$

$$\Big[= \mathbb{E}\Big[\sum_{n=0}^{\infty} I\{X_n = k,\ T_j(1) > n\} \,\Big|\, X_0 = j \Big]$$

Consider

$$\sum_{k \in S} r_k p_{k\ell} = \sum_{k \in S} \sum_{n=0}^{\infty} \mathrm{Prob}(X_n = k, T_j(1) \geq n \mid X_0 = j) \times \mathrm{Prob}(X_{n+1} = \ell \mid X_n = k)$$