02/12/04 Homogenous (stationary) Markov chains
- finite state, discrete time (FSDT)

Two ways to define a FSDT homogenous MC

1) Define an initial probability distribution $\vec{\phi}$

$$\phi_j = \text{Prob}(X_0 = j) \quad \text{for } j \in S$$

Define a probability transition matrix P

$$P_{ij} = \text{Prob}(X_{n+1} = j \mid X_n = i) \quad \text{for } i,j \in S$$

(Think of $S = \{0, 1, \ldots, N-1\}$ or $\{1, 2, \ldots, N\}$ or $\mathbb{Z}_N$)

2) Stochastic update rule:

$$X_{n+1} = f(X_n, Z_n)$$

with deterministic f and iid rvs $\{Z_n\}_{n=-\infty}^{\infty}$

and the initial probability distribution $\vec{\phi}$ for X_0.

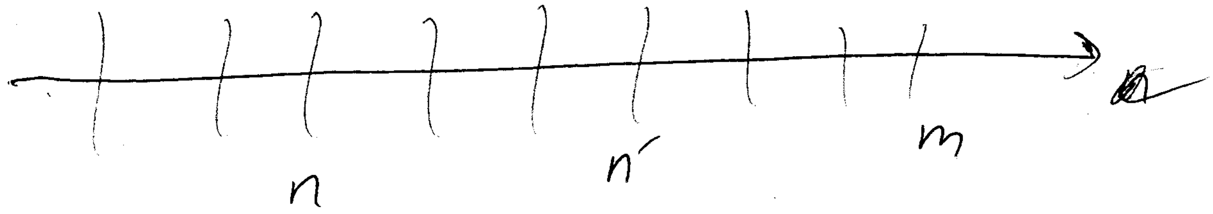
$$= \sum_{j' \in S} P_{j'j} \left(P^k\right)_{ij'} \quad \text{(induction hyp.)}$$

$$= \left(P^{k+1}\right)_{ij}$$

So we've proved $*$) for $k+1$.
This allows us to conclude $*$) is true for all $k \geq 1$.

We used in this proof:
Chapman-Kolmogorov equation:

$$\boxed{\begin{aligned} &\text{Prob}(X_m = j \mid X_n = i) = \\ &\quad \sum_{j' \in S} \text{Prob}(X_m = j \mid X_{n'} = j')\, \text{Prob}(X_{n'} = j' \mid X_n = i) \\ &\qquad \text{if } n < n' < m \end{aligned}}$$



- proved with same conditioning technique.

Why is it enough to specify $\vec{\Phi}$ and P to be able to calculate <u>all</u> statistics of a FSDT $\underset{\substack{\uparrow \\ \text{omogeneous}}}{H}$ MC?

1) Can calculate quantities:

$$*) \quad \text{Prob}(X_{n+k} = j \mid X_n = i) = (P^k)_{ij}$$

Proof (by induction):

$k=1$: clear

Assume $*)$ true for k. We'll prove it's true for $k+1$.

$$\text{Prob}(X_{n+k+1} = j \mid X_n = i)$$

$$= \sum_{j' \in S} \text{Prob}(X_{n+k+1} = j \text{ and } X_{n+k} = j' \mid X_n = i)$$

(mutually exclusive + exhaustive)

$$= \sum_{j' \in S} \underline{\text{Prob}(X_{n+k+1} = j \mid X_{n+k} = j', X_n = i)} \times \text{Prob}(X_{n+k} = j' \mid X_n = i)$$

$\left[\begin{array}{l} \text{conditional prob. rule:} \\ \quad \text{Prob}(A \text{ and } B \mid C) = \text{Prob}(A \mid B \text{ and } C) \\ \qquad\qquad\qquad\qquad\qquad \times \text{Prob}(B \mid C) \\ \quad \text{(no assumptions on } A, B, C \text{ other than measurable)} \end{array} \right]$

$$= \sum_{j' \in S} \underline{\text{Prob}(X_{n+k+1} = j \mid X_{n+k} = j')} \, \text{Prob}(X_{n+k} = j' \mid X_n = i)$$

(Markov property)

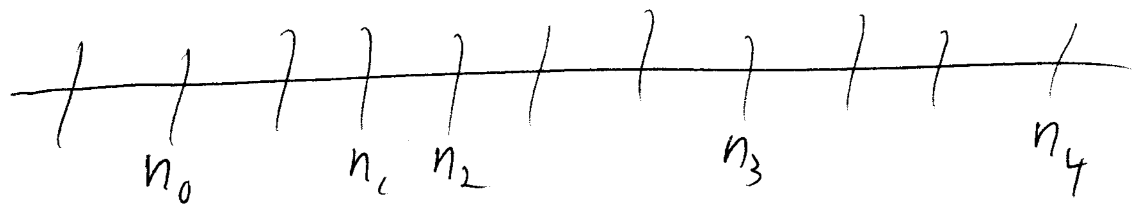
2) Can calculate quantities:

$$\text{Prob}(X_{n_k} = j_k \text{ for } 0 \le k \le K) \quad \text{with } (0 < n_0 < n_1 < n_2 < \cdots < n_K)$$

$$= \text{Prob}(X_{n_0} = j_0) \times \text{Prob}(X_{n_1} = j_1 \mid X_{n_0} = j_0) \times \cdots$$

$$\times \text{Prob}(X_{n_K} = j_K \mid X_{n_{K-1}} = j_{K-1})$$

by repeated use of Chapman-Kolmogorov equation



$$\text{Prob}(X_{n_0} = j_0) = \sum_{i \in S} \text{Prob}(X_{n_0} = j_0 \text{ and } X_0 = i)$$

(mutually exclusive + exhaustive)

$$= \sum_{i \in S} \text{Prob}(X_{n_0} = j_0 \mid X_0 = i)\, \text{Prob}(X_0 = i)$$

$$= \sum_{i \in S} (P^{n_0})_{i j_0}\, \phi_i$$

$$= (\vec{\phi} \cdot P^{n_0})_{j_0}$$

This shows you can calculate any finite dimensional dist from $\vec{\phi}$ and P.

Examples of FSDT HMC

A) Two-state on/off system:

At each busy time, probability q to become available at next time.

At each free time: probability p to become busy at next time

State 1: off/available/free

State 2: on/busy

$$\text{Specify } \vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad \begin{array}{l} \phi_1 \to \text{prob}(\overset{\text{free}}{\text{off}} \text{ at time } 0) \\ \phi_2 \to \text{prob}(\underset{\text{busy}}{\cancel{\text{on}}} \text{ at time } 0) \end{array}$$

$$P = \begin{array}{c c} & \begin{array}{cc} \text{free} & \text{busy} \end{array} \\ \begin{array}{c} \text{free} \\ \text{busy} \end{array} & \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array}$$

Fix this up!

$$X_{n+1} = \min\left((X_n + Z_n)_+ , M \right)$$

$$(x)_+ = \max(x, 0)$$

Probability Transition matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ \\ \\ M \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & & M \end{array} \\ \begin{bmatrix} 1-p & p & 0 & & & \\ q & 1-p-q & p & 0 & & 0 \\ 0 & q & 1-p-q & p & 0 & \\ & 0 & q & \ddots & \ddots & \ddots \\ & 0 & 0 & \ddots & 1-p-q & p \\ & & & 0 & q & 1-q \end{bmatrix} \end{array}$$

B) ~~Queuing~~ Queueing model with maximum capacity M

Assume server handles 1 request at a time.

One way to model the queue is to discretize time in some regular intervals such that it is very unlikely for more than one service request (arrival) or service completion (departure) per time step.

Let X_n be the ~~# service requests (cu~~ # customers in the queue at time n.

Let's suppose that in each time interval,

Prob p for a new arrival

Prob q for a departure

Prob $1-p-q$ for no change

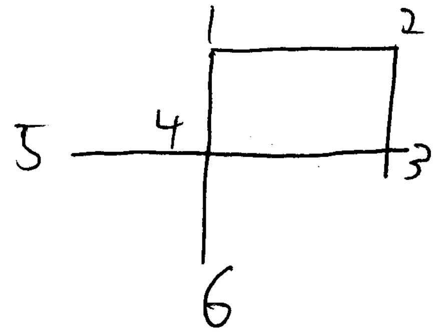
Stochastic update rule: $X_{n+1} = f(X_n, Z_n)$

$X_{n+1} = X_n + Z_n$ where Z_n is a r.v. with prob. dist. fn:

for $1 \le X_n \le M-1$

$$\text{Prob}(Z_n = 1) = p$$
$$\text{Prob}(Z_n = 0) = 1-p-q$$
$$\text{Prob}(Z_n = -1) = q$$

c) Random walk on a graph



Assume that from each location, one moves to an adjacent location with equal probability.

Discretize time by 1 time step/move

X_n = position of the random walker after nth move.

$$P = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 2 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 3 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 4 & 1/4 & 0 & 1/4 & 0 & 1/4 & 1/4 \\ 5 & 0 & 0 & 0 & 1 & 0 & 0 \\ 6 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

This was for random walks with "reflecting boundary conditions"

Can also have "absorbing" boundary conditions.

(Lawler Ch. 1)

d) Inventory model (Karlin + Taylor Ch. 2)

Max capacity M

Reordering policy: Reorder to go up to max capacity if the current inventory level falls below s. (new stock arrives next day in time to fill new orders)

Each day there is a certain amount of product demanded — immediately filled

Questions: average amount of inventory (over long run)
How much unfilled demand?
What fraction of time is there unsatisfied demand?

These can be addressed with a Markov chain model.

Discretize time in days (time between reordering decisions)

X_n = inventory level at end of day n after orders for that day are filled.

Model demand on day n as r.v. D_n

(i.i.d $\Rightarrow$ homogeneous/stationary)

A

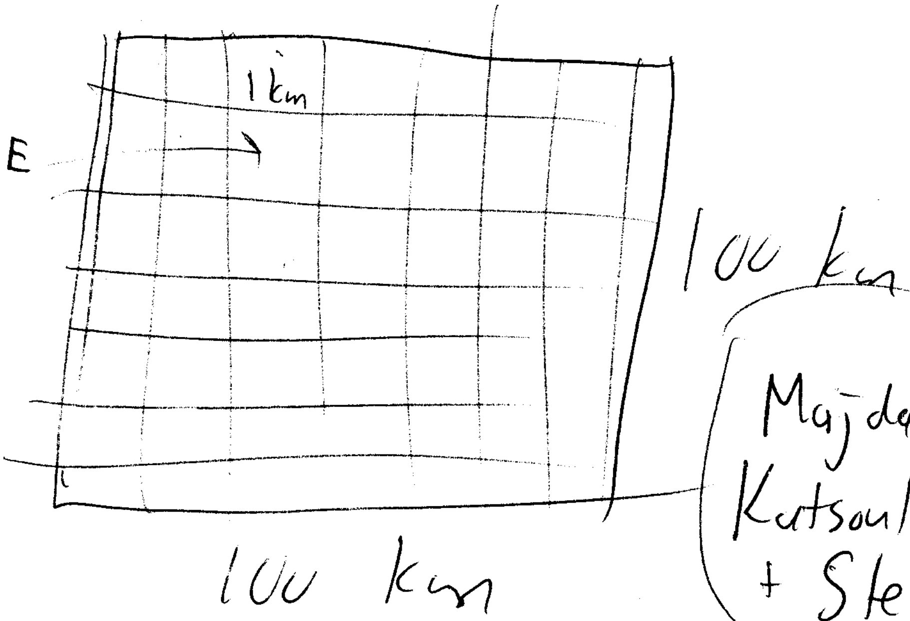
$$X_{n+1} = \begin{cases} (M - D_{n+1})_+ & \text{if } X_n \le s \\ (X_n - D_{n+1})_+ & \text{if } X_n > s \end{cases}$$

$$X_{n+1} = f(X_n, D_{n+1})$$

Probability transition matrix: somewhat complicated.

E) Simple MC model for tropical weather parameterization.

0 clear
1 critical CAPE
2 convection tower



Majda + Katsoulakis + Stevens + Sebesma

Dynamics