02/05/04 Simulation of random variables w/ continuous state space

Kloeden + Platen Sec. 1.3

General method: Inverse transform method

Take uniform distributed $U \sim U(0,1)$ $(0 \le U \le 1)$

$\sim$ distributed like

For general random variable with state space $S \subseteq \mathbb{R}$

$X_{sim} = F_X^{-1}(U)$ will have same prob. dist. as X.

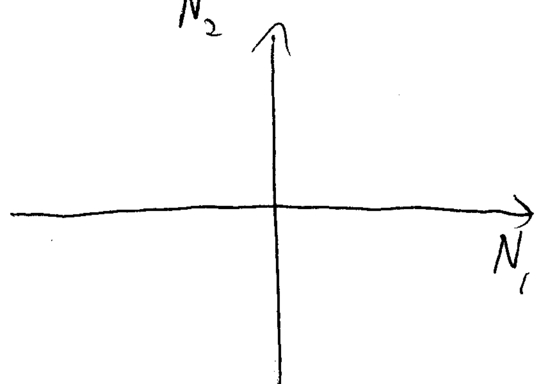
F_X: cdf

Quicker method for Gaussian random variables

Box-Müller method: To simulate Gaussian r.v. with mean 0 and variance 1 (standard Gaussian variable)

$X \sim N(0,1)$

0: mean, 1: variance



Better to simulate in pairs

Generate $U_1, U_2 \sim U(0,1)$

$$N_1 = \sqrt{-2\ln U_1}\,\cos 2\pi U_2$$

$$N_2 = \sqrt{-2\ln U_1}\,\sin 2\pi U_2$$

$N_1, N_2 \sim N(0,1)$ and independent.

Polar coordinates

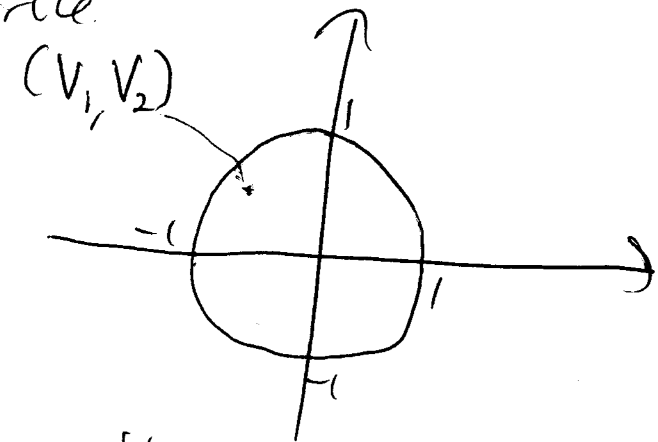
$\theta = 2\pi U_2$

$R = \sqrt{N_1^2 + N_2^2}$ has pdf

$p(r) \propto r e^{-r^2/2}$

Polar - Marsaglia Method: avoid trig evals

Generate a point uniformly distributed inside unit circle.



You can check that

$$R^2 = V_1^2 + V_2^2 \sim U(0,1)$$

$$\theta \sim U(0, 2\pi)$$

So $U_1 = R^2$, $U_2 = \frac{\theta}{2\pi} \sim U(0,1)$

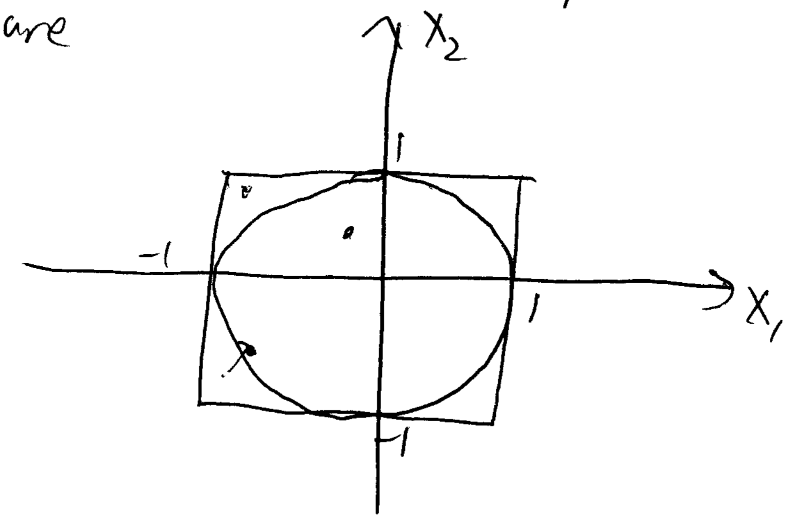
input into Box-Muller and note that $\cos 2\pi U_2 = \frac{V_1}{R}$ and $\sin 2\pi U_2 = \frac{V_2}{R}$

$$N_1 = \sqrt{\frac{-2 \ln R^2}{R^2}}\, V_1$$

$$N_2 = \sqrt{\frac{-2 \ln R^2}{R^2}}\, V_2$$

Rejection Method:

Simulate a uniformly distributed point in the square



$$X_1 = 2U_1 - 1 \qquad X_2 = 2U_2 - 1$$

with $U_1, U_2 \sim U(0,1)$

Then $X_1, X_2 \sim U(-1,1)$

To generate random point in circle:

Take the random point generated in the unit square if it's inside the circle.

If the point's outside the circle then throw it away and try again.

If can simulate $N_1 \sim N(0,1)$

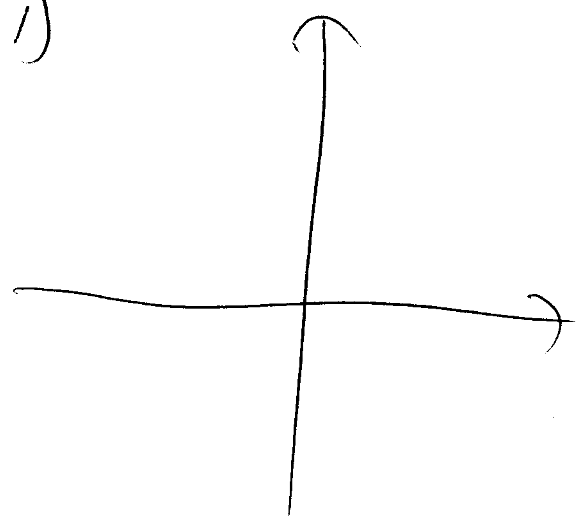
How can I get $X \sim N(\mu, \sigma^2)$

$$X = \mu + \sigma N_1$$

~~Another~~ How did we figure that, in the Box-Muller method,

$R = \sqrt{N_1^2 + N_2^2}$ has pdf $\propto r e^{-r^2/2}$

if $N_1, N_2 \sim N(0,1)$



Two ways:

1) Just rearrange $\text{Prob}(R \leq r)$

into $\text{Prob}(U_1 \leq g(r)) = g(r)$

as in proof of Inverse Transform Method

2) Use the rule for how a pdf changes under change of variables,

Let X have pdf $p_X(x)$

and $Y = f(X)$ have pdf $p_Y(y)$.

For now, suppose state spaces $S_X, S_Y \subseteq \mathbb{R}$.

and that f is one-to-one.

$$\text{Prob}(Y \in B) = \text{Prob}(f(X) \in B) = \text{Prob}(X \in f^{-1}(B))$$

for any $B \in \mathcal{B}_Y$

$$\text{Prob}(Y \in B) = \int_B p_Y(y)\,dy = \int_{f^{-1}(B)} p_Y(f(x))\,f'(x)\,dx$$

$$\text{Prob}(X \in f^{-1}(B)) = \int_{f^{-1}(B)} p_X(x)\,dx$$

This is true for any $B \in \mathcal{B}_Y$

$$p_Y(f(x))\,f'(x) = p_X(x)$$

$$\boxed{p_Y(y) = p_X(f^{-1}(y))\,(f^{-1})'(y)}$$ by inverse function theorem.

$x = f^{-1}(y)$

More generally:

If $S_{\underline{X}}, S_{\underline{Y}} \subseteq \mathbb{R}^n$ and f need not be 1-1.

$$\vec{\underline{Y}} = \vec{f}(\vec{\underline{X}})$$

$$p_{\underline{Y}}(\vec{y}) = \sum_{\vec{x} \in \vec{f}^{-1}(\vec{y})} p_{\underline{X}}(\vec{x}) \operatorname{Det}\left(\frac{\partial \vec{x}}{\partial \vec{y}}\right)$$

$\uparrow$

Jacobian

(Think: $p_{\underline{X}}\, d\vec{x} = p_{\underline{Y}}\, d\vec{y}$)

Gets a little trickier if Jacobian $= 0, \infty$.

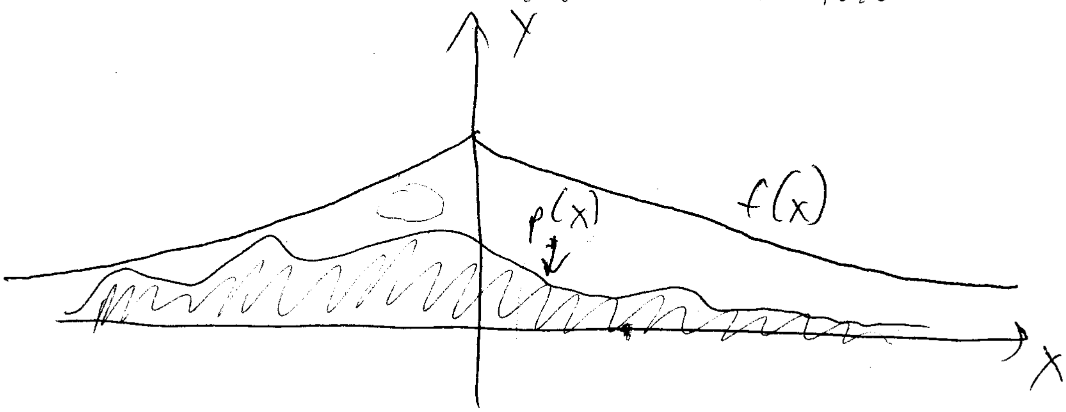
What about simulating other kinds of random variables?

State space $S \subseteq \mathbb{R}$.

Another general trick:

Rejection Method.

Suppose we have pdf $p(x)$ for X and inverse xform method too hard.



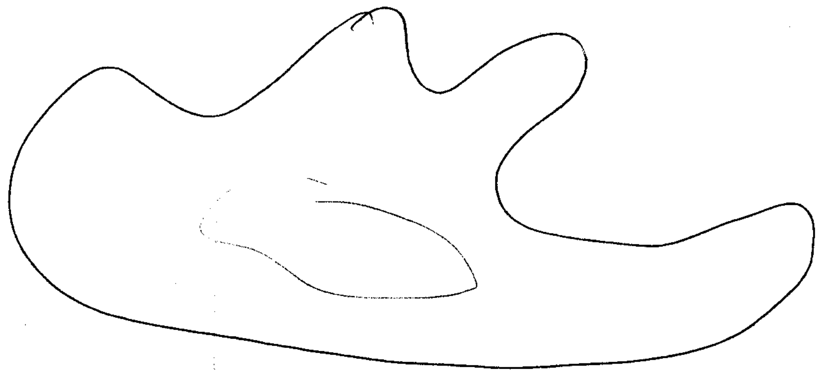
Notice that X can be simulated by picking a random point uniformly from the region $0 \le y \le p(x)$.

Now find a function $f(x)$ such that $0 \le p(x) \le f(x)$ and is easy to simulate a point uniformly below its graph.

(for example $f(x) \propto$ pdf for a random variable that's easy to simulate)

Generate uniform random points below $f(x)$ and reject if they're not below $p(x)$.

Uniform distribution in a region $R \subseteq \mathbb{R}^n$



For any $A \subseteq R$,

$$\text{Prob}(X \in A) = \frac{\text{Vol}(A)}{\text{Vol}(R)}$$

02/05/04 Generating functions and continuous ~~Gaussian~~ rv's

Usually don't use prob. gen. fn. for continuous rv's.

Characteristic function (moment generating fn)

$$\phi(k) = \langle e^{ikX} \rangle = \int_{-\infty}^{\infty} e^{ikx} p(x)\, dx$$

Cumulant generating function

$$\tilde{\phi}(k) = \ln \langle e^{ikX} \rangle$$

All the formulas same in discrete case and so random sums

$$Z = \sum_{j=1}^{N} X_j$$ with N discrete, X_j continuous,

formulas for $\langle Z \rangle$, $\langle (Z - \langle Z \rangle)^2 \rangle$ same as we calculated (using characteristic fns) for discrete case.

Cumulants ~~and Gaussian r.v.'s~~

$$\mu_n = \left(-i\frac{d}{dk}\right)^n \tilde{\phi}(k)\Big|_{k=0}$$

$$\parallel\!\!\mid$$

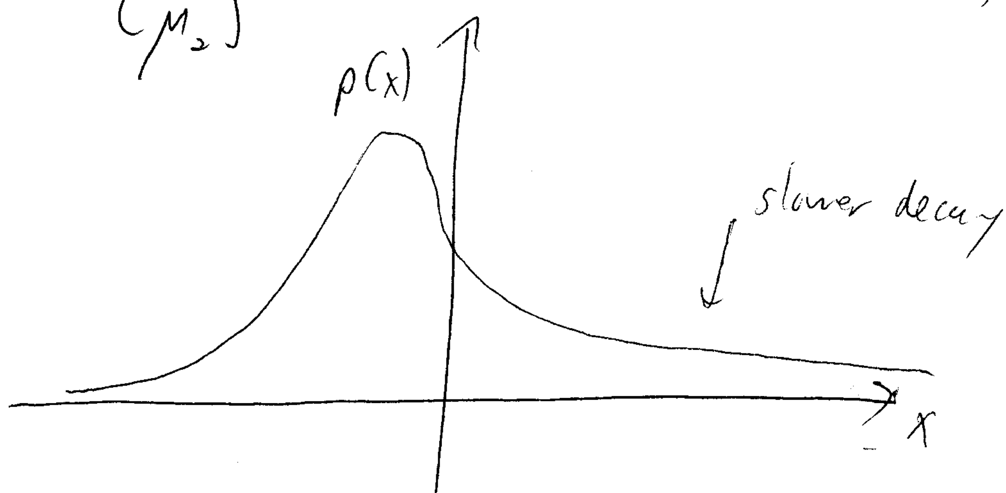
$$\langle\langle X^n \rangle\rangle$$

$$\mu_1 = \langle X \rangle$$

$$\mu_2 = \sigma_X^2$$

Third order cumulant reflects skewness of prob. distribution.

$$S = \frac{\mu_3}{(\mu_2)^{3/2}} \quad (= \text{skewness in some refs})$$



$$\mu_1 = 0$$

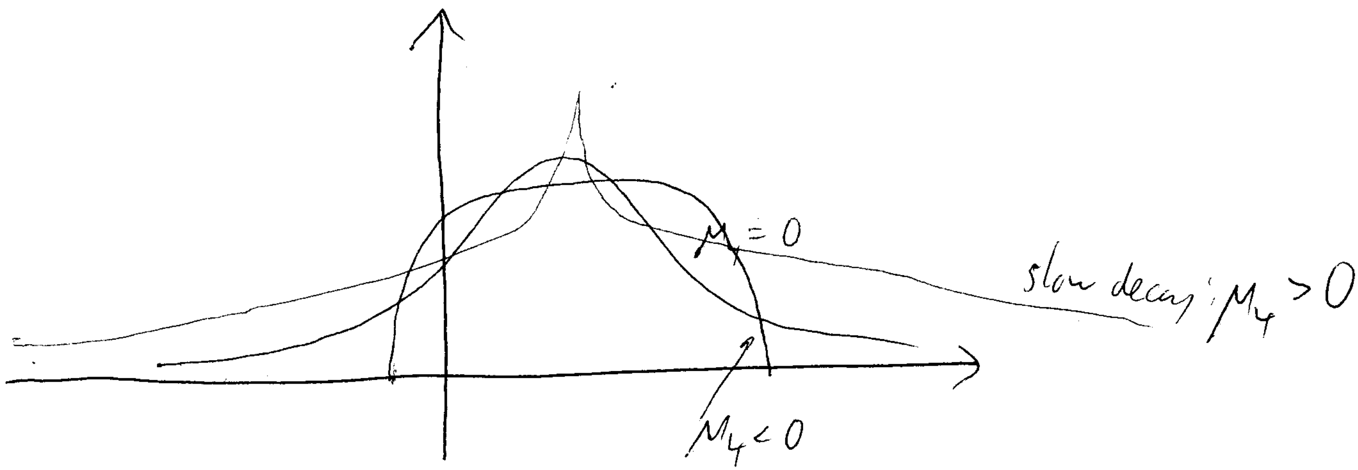
$$\mu_3 > 0$$

Fourth cumulant μ_4: "peakedness" of a distribution.

Kurtosis $= \dfrac{\mu_4}{\mu_2^2}$: only has to do w/ shape

(other defns are used)

Gaussian distribution has $\mu_4 = 0$.



Fat tails: large positive kurtosis, $\mu_4 > 0$.

Very thin or no tails: $\mu_4 < 0$.

- unlikely to have large fluctuations.

Cumulants + Gaussian r.v.s. $\boxed{\text{Risken Ch. 2}}$

$$p(x) = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$$ is pdf for $X \sim N(\mu, \sigma^2)$

$$\phi(k) = \int_{-\infty}^{\infty} e^{ikx} p(x)\,dx$$

$$= e^{ik\mu - \frac{1}{2}k^2\sigma^2}$$ is characteristic fn

Cumulant generating fn

$$\hat{\phi}(k) = \ln \phi(k) = ik\mu - \frac{1}{2}k^2\sigma^2$$

$$\mu_n = \left(-i\frac{d}{dk}\right)^n \hat{\phi}(k)\Big|_{k=0}$$

$\mu_1 = \mu$

$\mu_2 = \sigma^2$

$\mu_n = 0$ for $n \geq 3$

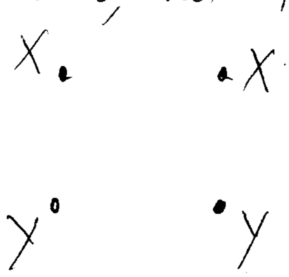
This is one reason why Gaussians are easy to calculate with.

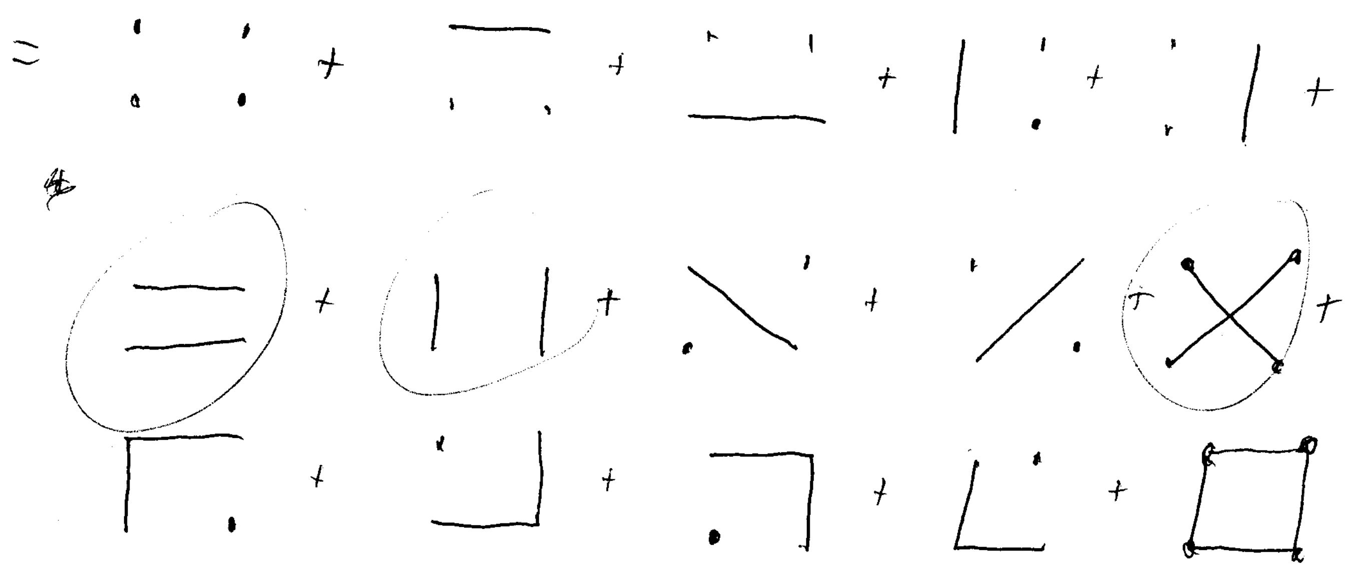
Wick's theorem

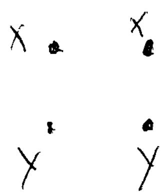
Diagrammatically: Cumulants are connected diagrams.
Moments are sums of all diagrams involving n points (n = order of moment).

Suppose X and Y are general rvs, not independent.

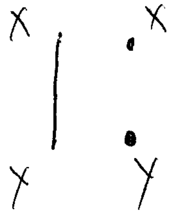
$\langle X^2 Y^2 \rangle$





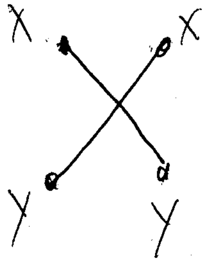


: $\mu_{X,1}\mu_{X,1}\mu_{Y,1}\mu_{Y,1}$



$$= \mu_{XY,2}\,\mu_{X,1}\,\mu_{Y,1}$$

$$\mu_{XY,2} = \langle\langle XY \rangle\rangle = \left(-i\frac{\partial}{\partial k_1}\right)\left(-i\frac{\partial}{\partial k_2}\right)\ln \langle e^{i(k_1 X + k_2 Y)} \rangle$$



$$= (\mu_{XY,2})^2$$

~~cases~~ If however X and Y are jointly distributed Gaussian r.v.'s, then <u>all</u> third and higher cumulants vanish

($\cancel{p(x,y) = \exp}$

$$p(\vec{x}) = \cancel{\frac{\exp\left(-\frac{1}{2}\vec{x}\cdot\mathcal{C}^{-1}\cdot\vec{x}\right)}{\sqrt{2\pi \det \mathcal{C}}}}$$

$$p(\vec{x}) = \frac{\exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu})^T\cdot \mathcal{C}^{-1}\cdot(\vec{x}-\vec{\mu})\right)}{\sqrt{2\pi \det \mathcal{C}}}$$

where $\vec{\mu} = \langle \vec{X} \rangle$

$$\mathcal{C}_{ij} = \langle (X_i - \mu_i)(X_j - \mu_j) \rangle$$