+ characteristic functions

Random variable X which takes values in state space $S = \mathbb{Z}_{\geq 0}$.

Probability generating function

$$G_X(s) = \langle s^X \rangle$$ defined for ~~[illegible]~~ $-1 \leq s \leq 1$.

Characteristic function

$$\phi_X(k) = \langle e^{ikX} \rangle$$ defined for all real k.

- this is also the "moment generating fn"

Cumulant generating fn

$$\hat{\phi}_X(k) = \ln \phi_X(k)$$

Application: Suppose I have a random variable

$$Z = \sum_{j=1}^{N} X_j$$

where N is random, Poisson distributed, with $\langle N \rangle = \lambda$.

and the $\{X_j\}$ are i.i.d. with

$$\text{Prob}(X_j = 1) = p$$
$$\text{Prob}(X_j = 0) = 1 - p$$

N is independent of the $\{X_j\}$

~~Direct~~ Question: What is the prob. dist. for Z?

Direct calculation:

$$\text{Prob}(N = n) = e^{-\lambda} \frac{\lambda^n}{n!}$$ because $\langle N \rangle = \lambda$, for $n \in \mathbb{Z}_{\geq 0}$.

$$\text{Prob}(Z=k) = \sum_{n=0}^{\infty} \text{Prob}(Z=k \text{ and } N=n)$$

(rule of marginal probability)

$$= \sum_{n=0}^{\infty} \text{Prob}(Z=k \mid N=n)\, \text{Prob}(N=n)$$

(rule of conditional probability)

$$\text{Prob}(A \text{ and } B) = \text{Prob}(A|B)\, \text{Prob}(B))$$

$$\text{Prob}(Z=k \mid N=n) \qquad Z = \sum_{j=1}^{n} X_j$$

$$= \underbrace{\binom{n}{k}}\, p^k (1-p)^{n-k} \qquad \text{for } 0 \le k \le n$$

$$= 0 \quad \text{for } k > n$$

$$\text{Prob}(Z=k) = \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{e^{-\lambda}\lambda^n}{n!}$$

$$= \sum_{n=k}^{\infty} \frac{\cancel{n!}}{(n-k)!\,k!} p^k (1-p)^{n-k} \frac{e^{-\lambda}\lambda^n}{\cancel{n!}}$$

$$= \frac{p^k e^{-\lambda}\lambda^k}{k!} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k}\lambda^{n-k}}{(n-k)!}$$

$$m = n-k$$

$$= \frac{p^k e^{-\lambda}\lambda^k}{k!} \sum_{m=0}^{\infty} \frac{(\lambda(1-p))^m}{m!}$$

$$= \frac{p^k e^{-\lambda} \lambda^k}{k!} \, e^{\lambda(1-p)}$$

$$\mathrm{Prob}(Z=k) = \frac{e^{-\lambda p}(\lambda p)^k}{k!}$$

Z is a Poisson r.v. with $\langle Z \rangle = \lambda p$

Now by generating functions.

- independent random vars appearing in discrete chunks.

$$G_Z(s) = \langle s^Z \rangle = \langle s^{\sum_{j=1}^N X_j} \rangle$$

$$= \langle \prod_{j=1}^N s^{X_j} \rangle$$

$$= \langle \langle \prod_{j=1}^N s^{X_j} \rangle_X \rangle_N$$

$\langle \cdot \rangle_X$: average over X vars

$\langle \cdot \rangle_N$: average over N vars.

$$= \langle \prod_{j=1}^N \langle s^{X_j} \rangle_X \rangle_N = \langle (G_X(s))^N \rangle_N$$

where $G_X(s)$ is the common prob. generating function for the $\{X_j\}$

$$G_Z(s) = G_N(G_X(s))$$

$$G_N(s) = \langle s^N \rangle$$ (Karlin + Taylor Sec. 1.1)

$$G_X(s) = \langle s^{X_j} \rangle = \sum_{n=0}^{\infty} \mathrm{Prob}(X_j = n)\, s^n$$

$$= (1-p)s^0 + ps^1 + 0s^2 + 0s^3 + \ldots$$

$$= 1 - p + ps$$

$$G_N(s) = \langle s^N \rangle = \sum_{n=0}^{\infty} \mathrm{Prob}(N = n)\, s^n$$

$$= \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} s^n$$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda s)^n}{n!} = e^{-\lambda} e^{\lambda s}$$

$$= e^{\lambda(s-1)}$$

Karlin + Taylor Sec. 1 has prob. gen fn. for various basic prob dists.

$$G_Z(s) = e^{\lambda(1-p+ps-1)} = e^{+\lambda p(s-1)}$$

which is gen fn. for Poisson rv with mean $\langle Z \rangle = \lambda p$.

Another excursion w/ gen fns. + char fns.

Random sum:

$$Z = \sum_{j=1}^{N} X_j \quad \text{with } N \text{ random}$$

X_j are i.i.d.

Question: How is mean and variance of Z related to those of the N and X_j?

We'll address using characteristic fn and prob. gen. fn.

Characteristic function (just because it generalizes to case of continuous X_j)

$$\phi_Z(k) = \langle e^{ikZ} \rangle = \langle e^{ik\sum_{j=1}^{N} X_j} \rangle$$

$$= \langle \langle e^{ik\sum_{j=1}^{N} X_j} \rangle_X \rangle_N$$

$$= \langle \prod_{j=1}^{N} \langle e^{ikX_j} \rangle_X \rangle_N$$

since the X_j are independent of each other and N.

$$= \langle (\phi_X(k))^N \rangle_N$$

where $\phi_X(k) = \langle e^{ikX_j} \rangle$ for all j (since identical di...

$$\phi_Z(k) = G_N(\phi_X(k)) \quad \text{with } G_N(s) = \langle s^N \rangle$$

Remember that ϕ is a moment generating function

$$\phi_Z(k) = \langle e^{ikZ} \rangle \qquad \phi_X(k) = \langle e^{ikX} \rangle$$

$$\langle Z^n \rangle = \left(-i \frac{d}{dk}\right)^n \phi_Z(k) \Big|_{k=0}$$

$$\langle Z \rangle = \left(-i \frac{d}{dk}\right) \phi_Z(k) \Big|_{k=0} = -i \phi_X'(k)\, G_N'(\phi_X(k)) \Big|_{k=0}$$

$$= -i \phi_X'(0)\, G_N'(\phi_X(0))$$

$$= -i \phi_X'(0)\, G_N'(1)$$

$$\langle Z^2 \rangle = \left(-i \frac{d}{dk}\right)^2 \phi_Z(k) \Big|_{k=0} = (-i \phi_X'(k))^2 G_N''(\phi_X(k)) \Big|_{k=0}$$

$$- \phi_X''(k)\, G_N'(\phi_X(k)) \Big|_{k=0}$$

$$= -(\phi_X'(0))^2 G_N''(1)$$

$$- \phi_X''(0)\, G_N'(1)$$

Relate derivatives of ϕ_X and G_N to moments of X and N

$$-i \phi_X'(0) = \langle X \rangle = \mu_X$$

$$-\phi_X''(0) = \langle X^2 \rangle = \sigma_X^2 + \mu_X^2$$

More mathematically:

$$\phi_Z(k) = \mathbb{E}\, e^{ikZ} = \mathbb{E}\, e^{ik\sum_{j=1}^{N} X_j}$$

$$= \mathbb{E}\left(\mathbb{E}\left[e^{ik\sum_{j=1}^{N} X_j} \,\middle|\, N\right]\right)$$

$$= \mathbb{E}\left(\mathbb{E}\left(\prod_{j=1}^{N} e^{ikX_j} \,\middle|\, N\right)\right)$$

$$= \mathbb{E}\left(\prod_{j=1}^{N} \mathbb{E}\left(e^{ikX_j} \,\middle|\, N\right)\right)$$

since the e^{ikX_j} are independent rv's,

$$= \mathbb{E}\left(\prod_{j=1}^{N} \mathbb{E}\, e^{ikX_j}\right)$$

$$= \mathbb{E}\left(\prod_{j=1}^{N} \phi_X(k)\right) = \mathbb{E}\left(\left(\phi_X(k)\right)^N\right)$$

$$= G_N\left(\phi_X(k)\right)$$

Be careful: G_N is not exactly a moment generating function.

What's true: $G_N(s) = \langle s^N \rangle = \langle e^{N \ln s} \rangle$

$$\langle N^n \rangle = \left(\frac{d}{d \ln s}\right)^n G_N(s)\Big|_{s=1} = \left(s \frac{d}{ds}\right)^n G_N(s)\Big|_{s=1}$$

Or: $G_N'(s) = \langle N s^{N-1} \rangle$

$G_N''(s) = \langle N(N-1) s^{N-2} \rangle$

$G_N'(1) = \langle N \rangle \stackrel{!}{=} \mu_N$

$G_N''(1) = \langle N(N-1) \rangle = \langle N^2 \rangle - \langle N \rangle$

$= \sigma_N^2 + \mu_N^2 - \mu_N$

Plug in the values for $\phi_X'(0), \phi_X''(0), G_N'(1), G_N''(1)$ into the expressions for $\langle Z \rangle, \langle Z^2 \rangle$

$\langle Z \rangle = \mu_X \mu_N$

$\sigma_Z^2 = \langle Z^2 \rangle - \langle Z \rangle^2 = \mu_X^2 \sigma_N^2 + \sigma_X^2 \mu_N$

Another problem for HW 1

- redo this calculation using cumulant gen fns for N, X,

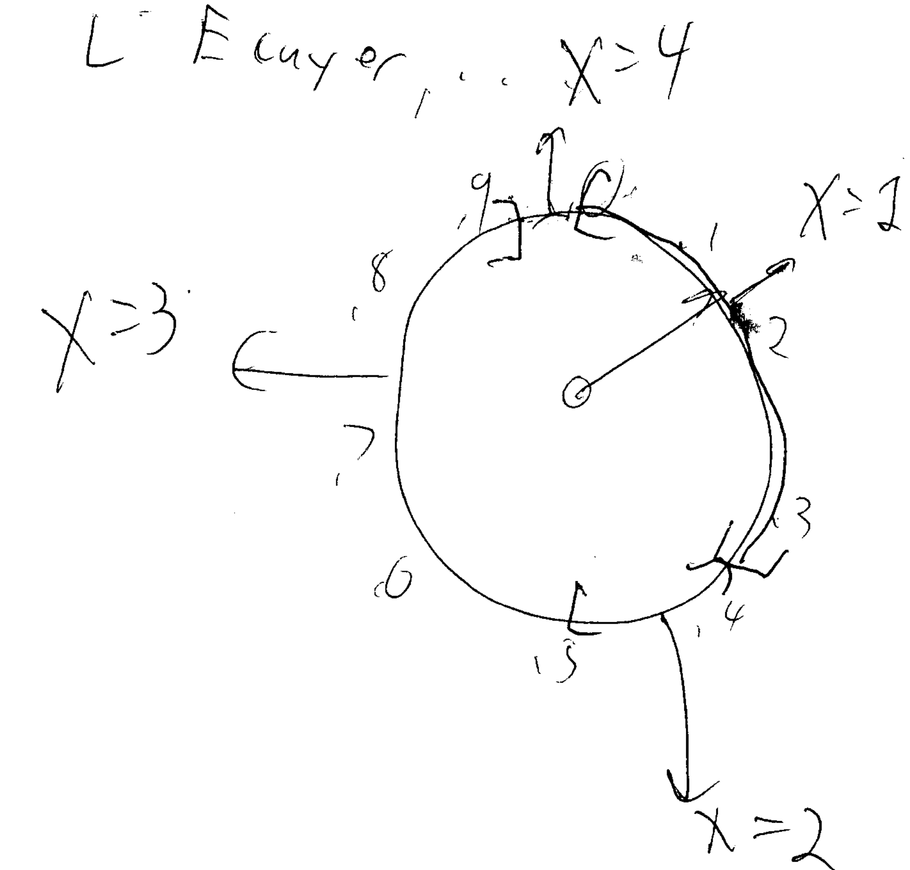
HW 1 now due Thursday 02/05/04 at 5 PM,

Simulating discrete random variables.

Finite state space S.

Computer simulates "pseudo-random" uniformly distributed random numbers
$0 \le x \le 1$.

L'Ecuyer, ...



$\text{Prob}(X=1) = .35$

$\text{Prob}(X=2) = .15$

$\text{Prob}(X=3) = .45$

$\text{Prob}(X=4) = .05$

Some other considerations:
If complicated, think of breaking up into simpler pieces.

Example: binomial dist

$$\text{Prob}(X=j) = \binom{N}{j} p^j (1-p)^{N-j}$$

$N = 1000$

Poisson random variables a pain to simulate this way

- but easy using stochastic process ideas!

What if the probability space or state space is uncountably big ~~(not discrete)~~.

- continuous random variable
- infinite # of discrete random variable.
- measure theory gets technical here.