Random variables

Consider random variable X which can take values from some discrete set S (state space)

range of r.v.

Can think of a random variable as a function mapping from the probability space Ω to the state space S

Quantify the random variable:

$$p_x = \text{Prob}(\underbrace{X = x}_{\substack{\text{This is an event} \\ (\text{a subset } A \subseteq \Omega)}}) \quad \text{for } x \in S$$

Note that we can in this way

Note that we can in this way define a probability measure on the state space S.

$$P_X(B) = \sum_{x \in B} p_x \quad \text{for } B \subseteq S.$$

This is working with S as a condensed probability space if all the info we care about is the value of the r.v. X.

Summarize the statistics of a r.v. X:

Mean: $\mu_X = \langle X \rangle = \mathbb{E} X$

$$\equiv \sum_{x \in S} x \, p_x$$

More generally, for any function f

$$\langle f(X) \rangle = \sum_{x \in S} f(x) \, p_x$$

$$\langle a f(X) + b g(X) \rangle = a \langle f(X) \rangle + b \langle g(X) \rangle$$

if a, b deterministic.

Variance: $\sigma_X^2 = \langle (X - \mu_X)^2 \rangle$

$$= \langle X^2 - 2\mu_X X + \mu_X^2 \rangle$$

$$= \langle X^2 \rangle - 2\mu_X \langle X \rangle + \mu_X^2$$

$$= \langle X^2 \rangle - 2\mu_X^2 + \mu_X^2$$

$$\sigma_X^2 = \langle X^2 \rangle - \mu_X^2$$

Standard deviation $\sigma_X = \sqrt{\sigma_X^2}$

- typical departure from mean.

Finer information:

Higher moments

$$\langle X^n \rangle = \sum_{x \in S} x^n p_x$$

Cumulants more useful way of organizing finer information

Examples of random variables
with a discrete state space

1) Binomial distribution:

$$S = \{0, 1, \ldots, N\}$$

~~Prob(X = a)~~

$$\text{Prob}(X = j) = \binom{N}{j} p^j (1-p)^{N-j}$$

for some number $0 \leq p < 1$

Interpretation: Bernoulli trials
- N "trials", each independent
- p = probability of 'success' on any one trial

X = # successes in N trials.

$$\binom{N}{j} = {}_N C_j = \text{"N choose j"}$$
$$= \frac{N!}{(N-j)!\, j!}$$

2) Uniform distribution

$$S = \{0, 1, \ldots, N\}$$

$$\text{Prob}(X = j) = \frac{1}{N+1} \quad \text{for } 0 \le j \le N$$

3) Poisson distribution

- sort of a generalization of uniform distribution to an ~~a~~ set

$$-S = \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$$

$$\text{Prob}(X = j) = e^{-\lambda} \frac{\lambda^j}{j!}$$

for some parameter $\lambda > 0$.

$$\langle X \rangle = \lambda \quad \text{(will show later)}$$

- models # events that occur in a certain space-time domain if these events occur with a constant but random intensity.