2-24-02

To solve $ay'' + by' + cy = g(t)$

Step 1 - find general solution of

$ay'' + by' + y = 0$, get $C_1 y_1(t) + C_2 y_2(t)$

Step 2 Method of undetermined coefficients

(if $g$ is polynomial, exponential, sin or cos

a) write down form of trial solution $Y(t)$

b) Plug in $Y(t)$ and solve for unknown coefficients

step 3 general solution of ✡ is

$C_1 y_1(t) + C_2 y_2(t) + Y(t)$

If $g(t)$ is

$$(a_0 t^n + a_1 t^{n+1} + \cdots + a_n) e^{\alpha t} \begin{cases} \sin\beta t \\ \text{or} \\ \cos\beta t \end{cases}$$

then try $Y(t)$ of the form

$$t^s [(A_0 t^n + A_1 t^{n-1} + \dots A_n) e^{\alpha t} \cos\beta t + (B_0 t^n + B_1 t^{n-1} + \dots B_n) e^{\alpha t} \sin\beta t]$$

where $s$ is the smallest integer that ensures that $Y(t)$ is not a solution of the homogeneous problem ($s = 0, 1, 2$)

Ex $y'' + 3y + 2y = 5t^2$

$$\left.\begin{array}{l} \text{Solve } y'' + 3y' + 2y = 0 \quad y = e^{rt} \\ r^2 + 3r + 2 = 0 \\ r = -1, -2 \\ y = C_1 e^{-t} + C_2 e^{-2t} \end{array}\right\} \text{Step 1}$$

Step 2: Try $Y = A + Bt + Ct^2$

$Y' = B + 2Ct$

Plug $Y'' = 2C$

$2C + 3(B + 2Ct) + 2(A + Bt + Ct^2) = 5t^2$

Collet powers of $t$

$t^2(2C) + t(2B + 6C) + (2C + 3B + 2A) = 5t^2$

coff of $t^2$: $2C = 5$ $\quad C = 5/2$

.... $t$: $2B + 6C = 0$ $\quad B = -15/2$

$\ldots 1: 2C + 3B + 2A = 0 \quad A = {}^{35}/_{4}$

$Y = {}^{35}/_{4} - {}^{15}/_{2}t + {}^{5}/_{2}{}^{2}$

$y = C_1 e^{-t} + C_2 e^{-2t} + {}^{35}/_{4} - {}^{15}/_{2}t + {}^{5}/_{2}t^2$

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Ex $L[y] = y'' + 4y = 1 + x + x^2 e^x$

How do we find $Y$?

Split into 2 problems

$L[Y_1] = 1+x \qquad L[Y_2] = x^2 e^x$

then $Y = Y_1 + Y_2$

$L[Y] = L[Y_1 + Y_2] = L[Y_1] + L[Y_2]$

$= 1 + x + x^2 e^x$

guess

$Y_1 = Ax + B$

guess

$Y_2 = (C + Dx + Ex^2)e^x$

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$\boxed{\text{Ex}}$ $[-y'' + y' = \cos \omega t]$

$\text{Re}[-Y'' + Y' = e^{i\omega t}]$

$$-\underbrace{(\text{Re}\,Y)''}_{y} + \underbrace{(\text{Re}\,Y)'}_{y} = \underbrace{\text{Re}(e^{i\omega t})}_{\cos \omega t}$$

If the forcing term is $\cos \omega t$, consider instead the problem with $\cos \omega t$ replaced by $e^{i\omega t}$, solve that problem to get $Y$. Then $y = \text{Re}[Y]$ solves the original problem with $\cos \omega t$ forcing term.

$-Y'' + Y' = e^{i\omega t}$

Try $Y = Ae^{i\omega t}$

complex

Try $Y = Ae^{i\omega t}$

complex

$Y' = i\omega Ae^{i\omega t}$

$Y'' = (i\omega)^2 Ae^{i\omega t}$

$= -\omega^2 Ae^{i\omega t}$

plug in: $\omega^2 A \cancel{e^{i\omega t}} + i\omega \cancel{Ae^{i\omega t}} = \cancel{e^{i\omega t}}$

$A(\omega^2 + i\omega) = 1 \Rightarrow A = {}^{1}/_{\omega^2 + i\omega}$

transfer function

$$Y = \frac{1}{\omega^2 + i\omega} e^{i\omega t}$$

$y = \mathrm{Re}[Y]$

$= \frac{1}{\omega^2 + i\omega}\left(\frac{\omega^2 - i\omega}{\omega^2 - i\omega}\right) \cos\omega t + i \sin\omega t$

$\omega^4 + \omega^2$

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Our books method: Try $y = A\cos\omega t +$

$y' = -A\omega \sin\omega t + B\omega\cos\omega t$

$y'' = -A\omega^2 \cos\omega t - B\omega^2 \sin\omega t$

$A\omega^2\cos\omega t + B\omega^2 \sin\omega t - A\omega \sin\omega t + B\omega\cos\omega t = \cos\omega t$

cos coeff: $A\omega^2 + B\omega = 1$

sin coeff: $B\omega^2 - A\omega = 0$ $\quad A = B\omega$

$B\omega^2 + B\omega = 1$

$B = \frac{1}{\omega^2 + \omega}$

$y = \frac{\omega}{\omega^3 + \omega}\cos\omega t + \frac{1}{\omega^3 + \omega}\sin\omega t$