Exam Wed. Feb 11
Extra office hrs Tuesday Feb 10 4 - 5

Topics
First-order ODE's: general sol, solve initial value problem
  - graphical methods
  - autonomous equations
  - analytical methods
    - integrating factors
    - separation of variables
  - numerical method: Euler's method
Modeling problems
Second-order ODE's: general sol, solve initial value problems
  - constant-coeff. linear homogeneous eqs.

8.3.2 continued

Last time: linear independence of functions

\[ f_1, f_2 \text{ are lin. indep. if } k_1 f(t) + k_2 g(t) = 0 \]

\[ f_1' = f_2' \]

\[ k_1 + k_2 = 0 \]

\[ W(f_1, f_2)(t_0) \neq 0 \rightarrow f_2 \text{ lin. indep. for some } t_0 \]

Thm. 8.3.2 (Abel's Thm)

If \( y_1, y_2 \text{ solve } y'' + p(t)y' + q(t)y = 0 \)

then \( W(y_1, y_2) \) is always 0 or never zero

\[ \begin{align*}
  y_1'y_2'' - y_2'y_1'' + p(y_1'y_2 - y_2'y_1) &= 0 \\
  W'(y_1, y_2) + W(y_1, y_2) &= 0
\end{align*} \]

This is a first order, linear ODE for \( W \).
Solve by method of integrating factor ... get \( W = ce^{-\int p(t) dt} \).
Note that to compute the Wronskian (up to the constant) we only need to know the ODE not its solutions.

This formula for the \( W = C e^{-\int p \, dt} \) also gives us a way to find a second solution to the ODE if we already know one solution.

\[
W \, \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W = C e^{-\int p \, dt}
\]

If \( y_1 \) is known then this is a linear 1st order ODE for \( y_2 \).

The book uses method of reduction of order

look for \( y_2 = v(t) y_1(t) \)

For solutions of \( y'' + p(t)y' + q(t)y = 0 \) the following is equivalent:

1. \( \{y_1, y_2\} \) form a fundamental set of solutions
2. \( y_1 \) and \( y_2 \) are linearly independent
3. \( W(y_1, y_2)(t_0) \neq 0 \) for some \( t_0 \)
4. \( W(y_1, y_2)(t) \neq 0 \) for all \( t \)

To get a general solution of \( y'' + p(t)y' + q(t)y = 0 \),

find \( y_1 \) and \( y_2 \) satisfy

calculate \( W(y_1, y_2) \) is non-zero somewhere

If this is true, then the general solution of \( y'' + p(t)y' + q(t)y = 0 \) is \( c_1 y_1(t) + c_2 y_2(t) \)

**Vector Space Interpretation**

Solutions to 2nd order linear ODE are vectors in \( \mathbb{R}^2 \)

Linear independence

\( k_1 y_1 + k_2 y_2 = 0 \) \( \Rightarrow \) \( k_1 = k_2 = 0 \)

Linear combination of solutions is again a solution

Linear combination of two vectors is a vector

Fundamental set of solutions is a basis (\( \{1, \sqrt{3}\} \) for example)

**
Ex \( \{ e^{at}, e^{-at} \} \)
\[
\frac{e^{at} + e^{-at}}{2} \cdot \frac{e^{at} - e^{-at}}{2}
\]
\( \{ \cosh at, \sinh at \} \)

Any solution can be written as a linear combo of basis solutions (fundamental set).

Any vector in \( \mathbb{R}^2 \) can be written as a linear combo of vectors in \( \mathbb{R}^2 \).

Dimension of space of solution is the # of elements in fundamental set \( (= \text{order of ODE}) \)

Dimension of space = number of elements in basis