Backward Euler method

\[
\frac{dy}{dt} = f(t, y) \\
y(t_n) - y(t_{n-1}) \approx \frac{dy}{dt} \bigg|_{t_n} = f(t_n, y(t_n))
\]

Solve for \(y(t_n)\)

tends to be more stable

2.5 Population Growth and graphical methods for autonomous equations

First order nonlinear ODE

\[
\frac{dy}{dx} = f(y)
\]

Independent variable does not appear explicitly

Model 1: Exponential Growth

growth rate \& number of individuals \(\cdot y(t)\)

\[
y'(t) = ry \\
y(t) = y_0 e^{rt}
\]

Model 2: Logistic Growth

\[
\frac{dy}{dt} = f(y) y \\
f(y) \approx r \text{ for small populations} \\
f(y) \text{ decreases as population increases} \\
f(y) \text{ negative for large populations}
\]
\[ \frac{dy}{dt} = (r-ay)y \quad \text{Logistic eq.} \]

\[ \frac{dy}{dt} = r \left(1 - \frac{y}{k}\right) y \quad \text{separate variables} \]

\[ \left(1 - \frac{y}{k}\right) y = r \, dt \]

\[ \int \left(1 - \frac{y}{k} + \frac{\beta}{y}\right) dy \]

where \( \frac{dy}{dt} = 0 \), \( y \) does change.

\( y = 0 \), \( y = k \) \( \Rightarrow \) \text{equilibrium solutions}

where \( \frac{dy}{dt} > 0 \), \( y \) is increasing

where \( \frac{dy}{dt} < 0 \), \( y \) is decreasing
\[ \frac{dy}{dt} = 2(y - 1) + 0.5 = 2.85 \]

\[ t_2 = t_1 + h = 0.1 \]

\[ y_2 = y_1 + \frac{dy}{dt} |_{t_1} h = 4.1 + (2.85)(0.1) = 4.285 \]

**Graphical methods for autonomous first-order ODE**

\[ \frac{dy}{dt} = r \left( 1 - \frac{y}{k} \right) y \]

\[ f(y) \]

1. Graph \( f(y) \) as a function of \( y \)
2. Find the equilibrium pts. (where \( f(y) = 0 \))
3. Where \( f(y) \) is positive draw arrow to right and vice versa.
   (The arrows show how \( y \) changes with time)
4. Draw new graph of \( y \) as a function of \( t \).
   - Where arrow points to right solution has positive slope and vice versa.

**Classification of equilibrium pts**

- If arrows on both sides of an equilibrium pt. point towards the equilibrium pt. \( (\quad \rightarrow \quad) \), the equilibrium pt. is asymptotically stable.
- If arrows on both sides of an equilibrium pt. point away from the equilibrium pt. \( (\quad \leftarrow \quad \rightarrow \quad) \) the equilibrium pt. is asymptotically unstable.
If one arrow points towards the equilibrium $p_t$ and one away, the equilibrium $p_t$ is semi-stable.

<table>
<thead>
<tr>
<th>$rac{dy}{dt} = f(y)$</th>
<th>$\frac{df}{dy}$</th>
<th>$y(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+$</td>
<td>$+$ $\frac{dy}{dt}$ increasing with $y$</td>
<td>concave up</td>
</tr>
<tr>
<td>$+$</td>
<td>$-$ $\frac{dy}{dt}$ decreasing with $y$</td>
<td>concave down</td>
</tr>
<tr>
<td>$&lt;$</td>
<td>$+$ $\frac{dy}{dt}$ increasing as $y$ increases</td>
<td>concave down</td>
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<td>$&lt;$</td>
<td>$-$ $\frac{dy}{dt}$ decreasing as $y$ increases</td>
<td>concave up</td>
</tr>
</tbody>
</table>
Ex. \( \frac{dy}{dt} = -r \left( 1 - \frac{y}{T} \right) y \)

Equilibria: \( y = 0 \), \( y = T \)

Stable unstable

Ex. \( \frac{dy}{dt} = -r \left( 1 - \frac{y}{T} \right) \left( 1 - \frac{y}{K} \right) y \)

\( 0 < T < K \)

Equilibrium points: \( y = 0, T, K \)

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