Mixing problems

Do not work with concentrations; instead work with amount of salt given initial concentration.

\[ Q \text{ = amount of salt in the reservoir} \]

\[ \frac{dQ}{dt} \Rightarrow \left( \text{rate of salt flowing in} \right) - \left( \text{rate of salt flowing out} \right) \]

\[ \frac{dQ}{dt} \Rightarrow \left( \text{Incoming concentration} \times \text{Flow rate} \right) - \left( \frac{Q}{V} \right) \times \text{Outgoing flow rate} \]
Gas mixing in lungs

Initial concentration
70% N₂
30% O₂

How long does it take for the N₂ concentration to get below 10%?

Let

\[ v(t) = \text{volume of O₂ in lung at time } t \]

\[
\frac{dv}{dt} = \text{(rate at which volume of O₂ flowing in)} - \text{(rate at which it flows out)}
\]

\[
= \left( \frac{100\% \text{ or } 1}{6 \text{ l/min}} \right) (\text{flow rate}) - \left( \frac{v}{3 \text{ l}} \right) \left( \text{flow rate out} \right)
\]

\[
\frac{dv}{dt} = 6 - 2v
\]

\[
v(0) = 30\% \cdot 3 \text{ l} = 0.9 \text{ l}
\]

Use initial condition
\[ a = 3 + c \]
\[ c = -2.1 \]

\[ v = 3 - 2.1 e^{-2t} \]

We want to know what's \( t \) when \( v \) is 90% \( 3 \text{ l} = 2.7 \)

\[ 2.7 = 3 - 2.1 e^{-2t} \]

\[ 2.1 = e^{-2t} \]

\[ t = -\frac{\ln 7}{2} = 0.97 \text{ secs} \]
Mixing Problems

Need to know

1. How much volume of liquid is in the container at all time.
   (Usually not always the amount that goes in = out)
   (then \( \Rightarrow \) Volume = constant)

2. Diff Eq
   
   \[ \frac{dQ}{dt} = \text{Amount In} - \text{Amount Out} \]

   \[ \text{Amount In} = \left( \text{concentration} \right) \left( \text{Rate of incoming} \right) \]

   \[ \text{Amount Out} = \left( \frac{Q}{\text{Volume}} \right) \left( \text{Rate of outgoing} \right) \]

Example:

Container of Pure water at \( t=0 \), we add a salt solution \( (50/\text{L}) \) at a rate of \( 10/\text{min} \). Water exits the container at a rate of \( 15/\text{min} \). If there is 5 liters initially in the container. How much salt is in the container after 10 mins.

\[ Q = \text{amount of salt in tank} \]

\[ \frac{dQ}{dt} = \left( 10/\text{min} \right) \left( 50/\text{L} \right) - \left( 15/\text{min} \right) \left( \frac{Q}{5} \right) \]

\[ \frac{dQ}{dt} = 5 - \frac{Q}{5} \]

Pure water \( \Rightarrow Q(0) = 0 \Rightarrow \) net salt

First order, linear

\[ Q = 2S + Ce^{-\frac{t}{5}} \]

\[ Q(0) = 2S + C = 0 \Rightarrow C = -2S \]

\[ Q = 2S - 2Se^{-\frac{t}{5}} \]

At any time you can tell how much salt in the tank.

\[ Q(10) = 2S - 25e^{-2} \text{ (g of salt)} \]
What happens if we change the rate out? $r_{\text{out}} = \frac{1}{2}$ L/min? $r_{\text{in}} \neq r_{\text{out}}$

Volume increasing $\frac{1}{2}$ L/min

Volume in tank $= 5 + \frac{1}{2}t$

Rate of salt leaving $= \text{(concentration)} \times \text{(rate out)}$

$$\frac{dQ}{dt} = 5 - \left( \frac{Q}{10 + t} \right)$$

$Q(0) = 0$

1st order linear

$$\frac{dQ}{dt} + \frac{Q}{10 + t} = 5$$

$$\mu(t) = e^{\int \frac{1}{10 + t} \, dt} = 10 + t$$

$Q(10 + t) = 50t + \frac{5}{2}t^2 + C$

$Q = \frac{50t + \frac{5}{2}t^2 + C}{10 + t}$

$Q(0) = \frac{C}{10} = 0 \quad C = 0$

$Q = \frac{50t + \frac{5}{2}t^2}{10 + t}$

Amount of salt in tank at any time.
When will there be 125 lbs salt in the tank.

\( Q(t) \) = amount of salt

\[
\frac{dQ}{dt} = (2 \text{ gal/min}) (3 \text{ lbs/salt}) - \left( 2 \frac{\text{gal}}{\text{min}} \right) \left( \frac{Q}{50 \text{ gal}} \right)
\]

\[
Q = 50 - \frac{Q}{25}
\]

\[
Q' + \frac{Q}{25} = 46
\]

\[
w(x) = e^{-\frac{x}{25}}
\]

\[
Q e^{\frac{x}{25}} = \int -6 e^{\frac{x}{25}} dx
\]

\[
Q e^{\frac{x}{25}} = -150 e^{\frac{x}{25}} + C
\]

\[
Q = -150 + Ce^{-\frac{x}{25}}
\]

Plug in 125 for Q solve for t

\[
t = 1
\]

Solve for C

\[
C = \frac{125}{e^{\frac{t}{25}}}
\]

Plug in Q and C and solve for t

\[
t = 0
\]
Quiz on Thursday
- no crib sheets
- separable
- first order linear

\[
\int e^x \sin x \, dx = e^x \sin (x) - \int e^x \cos (x) \, dx = e^x \sin x - e^x \cos x + \int e^x \sin (x) \, dx
\]

\[
\begin{align*}
\text{Let} & \quad u = \cos x & \quad du &= -\sin x \, dx \\
\text{u} &= \cos x & \quad u &= \cos x \\
\text{dv} &= e^x \, dx & \quad v &= \sin x \\
\text{v} &= \sin x & \quad v &= e^x \\
\end{align*}
\]

\[
\int e^x \sin (x) \, dx = e^x \sin (x) - e^x \cos (x) + 2 \int e^x \sin (x) \, dx
\]

\[
\int e^x \sin x \, dx = \frac{e^x \sin (x) - e^x \cos (x)}{2} + C
\]

**Mixing Equation**

A tank contains 5 L of fresh water. A saline solution is added to the tank that adds \(10^3 \text{ g/min}\). Water leaves the tank at the same rate it enters the tank (\(2^3/\text{min}\)). How much salt is in the tank at any time?

**Volume in Tank**

\(V = 5 \text{ Leters}\)

**Diff Eq for Salt**

\[
\frac{dQ}{dt} = 10^3 - \frac{Q}{5} \times 2
\]

\[
\frac{dQ}{dt} = 10 - \frac{2}{5} Q
\]

\(Q(0) = 0\)

\[
1^\text{st order linear} \quad \frac{dQ}{dt} + \frac{2}{5} Q = 10
\]

\[
\mu = e^{-\frac{2}{5}t}
\]

\[
Q = 2S - 2Se^{-\frac{2}{5}t}
\]

\(Q(e^{\frac{2}{5}t}) = 10e^{\frac{2}{5}t} + Ce^{-\frac{2}{5}t}
\]

\(Q(0) = 2S + C = 0\)

\(\Rightarrow C = -2S\)