$u_t = \frac{d^2 u}{dx^2}$
$u(0,t) = 0 = u(L,t)$
$u(x,0) = f(x)$

Step A: Find many solutions of $u_t = \frac{d^2 u}{dx^2}$

A.1 Try $u = X(x)T(t)$

separate variables

A.2 Solve $X$ eq.: $X'' + \lambda X = 0$

$X(0) = 0 = X(L)$

get $\lambda = \left( \frac{n\pi}{L} \right)^2$

$X(x) = \sin \frac{n\pi x}{L}$

A.3 Solve $T$ eq.: $T' + \alpha^2 \lambda T = 0$

get $T/T = Ce^{-\frac{\alpha^2 (n\pi)^2}{L} t}$

A.4 $u_n(x,t) = \sin \left( \frac{n\pi x}{L} \right) e^{-\alpha^2 \left( \frac{n\pi}{L} \right)^2 t}$

Step B: $u(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} e^{-\alpha^2 \left( \frac{n\pi}{L} \right)^2 t}$

Use initial conditions, get $C_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$

Note: $u(x,t) = \sum C_n \sin \frac{n\pi x}{L} e^{-\alpha^2 \left( \frac{n\pi}{L} \right)^2 t} \rightarrow 0$ as $t \rightarrow \infty$

$\sum$ convergence of sum

$u(x,t)$ is smooth for all $t > 0$
$f(x) = u(x,0)$

$\partial u/\partial x = u(x,0)$

$u(x,small t)$

$u(x,t)$
Other Boundary Conditions

\[ U_t = \alpha^2 U_{xx} \]
\[ u(0,t) = T_1, \quad u(L,t) = T_2 \]
\[ u(x,0) = f(x) \]

look at large time
If large-time solution is independent of \( t \)
\[ \phi^0 \to U_t = \alpha^2 U_{xx} \]
\[ u(\partial_s t) = 0 = (\delta L, 0) \]

For large \( t \), \( u(x,t) \to V(x) \)
Plug in \( u(t,x) = V(x) \) in to eq
\[ U_t = \alpha^2 U_{xx} = \alpha^2 V_{xx} \]

\[ 0 = V_t \]
\[ U(0,t) = 0 = u(L,t) \]
\[ V(0) = V(L) \]
\[ V_{xx} = 0 \]
\[ V(0) = 0 = V(L) \]
\( \Rightarrow \) \( V \) is identically 0

For the boundary conditions
\[ u(0,t) = T_1 \]
\[ u(L,t) = T_2 \]

Assume \( u(x,t) \to V(x) \)

\[ V_{xx} = 0 \]
\[ V(0) = T_1, \quad V(L) = T_2 \]

\[ V(x) = Ax + B \]
\[ T_1 = V(0) = B \]
\[ T_2 = V(L) = AL + T_1 \]
\[ A = \frac{T_2 - T_1}{L} \]

Write \( u(x,t) = V(x) + w(x,t) \)

develop steady-state solution
\[ V_{xx} = 0, \quad V(0) = T_1, \quad V(L) = T_2 \]
\[ V(x) = \frac{T_2 - T_1}{L} x + T_2 \]
What equation do \( w \) satisfy?
\[
\frac{\partial^2 w}{\partial t^2} + \alpha^2 \left( \frac{\partial^4 w}{\partial x^4} \right) = 0
\]
\[
\frac{\partial w}{\partial x} + w(0,t) = T_1
\]
\[
\frac{\partial w}{\partial x} + w(L,t) = T_2
\]
\[
V(x) + w(x,0) = f(x)
\]
\[
w(x,0) = f(x) - \left[ \frac{T_2 - T_1}{L} x + T_1 \right]
\]
\[
w_t = \alpha^2 \frac{\partial w}{\partial x}
\]
\[
w(0,t) = 0
\]
\[
w(L,t) = -\frac{T_2 - T_1}{L} L
\]

Apply method of separation of variables
\[
V(x,t) = \sum C_n \sin \frac{n\pi x}{L} e^{-\alpha^2 \left( \frac{n\pi}{L} \right)^2 t}
\]

\[
C_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx
\]

\[
V(x,t) = \frac{T_2 - T_1}{L} x + T_1 + \sum C_n \sin \frac{n\pi x}{L} e^{-\alpha^2 \left( \frac{n\pi}{L} \right)^2 t}
\]

steady state solutions

transient

Ex: rod of length 1 insulated on sides

\[
U_t = U_{xx}
\]
\[
U(0,t) = 10^\circ, \quad U(L,t) = 20^\circ
\]
\[
U(x,0) = \begin{cases} 10^\circ & 0 < x < \frac{L}{2} \\ 20^\circ & \frac{L}{2} < x < L \end{cases}
\]

Steady-state problem \( u(x,t) \to V(x) \) \( t \to \infty \)

\( V(0) = 10^\circ, \quad V(1) = 20^\circ \)

\[
V(x) = \frac{T_2 - T_1}{L} x + T_1 \quad \Rightarrow \quad V(x) = \frac{20 - 10}{1} x + 10^\circ = 10x + 10^\circ
\]

\( V(x) = a + bx \)

\( V(x) = 10x + 10^\circ \)
\[ 10 = v(0) = a \]
\[ 20 = v(1) = a + b = 10 + b \implies b = 10 \]
\[ v(x) = 10 + 10x \]

\[ u(x, t) = v(x) + w(x, t) \]

Plugging into eq.

Find eq. satisfied by \( w \)