Test 2  4-7-04

10.5 Partial differential equations
heat equation - diffusive behavior
wave equation - oscillatory behavior
Laplace's eq - steady

1st & 2nd order linear ODE's studied ODE's equation solutions
Fourier series
2 pt boundary value problems

PDE's "separation of variables"

earlier initial value problems $y(t_0) = y_0 \quad y'(t_0) = y_1 \quad t_0 \quad x$

$y'' + p(t)y' + q(t)y = g(t)$

$y(x) = y_0 \quad y_{(x)} = y_1$

$2 pt$ boundary value problem

or

$y'(x) = y_0 \quad y'(x) = y_1$

2 pt boundary value problem

$y(x) = y_0 \quad y'(x) = y_1$

can have no solution
1 solution
many solutions

We're interested in equation like: $x'' + \lambda x = 0$

$0 \quad L \quad x(0) = 0 = x(L)$
Ex. \[ x'' + \lambda x = 0 \]
\[ x'(0) = 0 = x'(L) \]

Case \( \lambda = 0 \) \[ x'' = 0 \]

Try \( x = e^{\lambda x} \)
\[ \lambda^2 = 0 \]
\[ e^0 = 1 \]

Solution \( x(x) = a x + b \)
\[ x'(x) = a \]
A non-zero solution is
\[ 0 = x'(0) = a \]
\[ x(x) = b \]
\[ 0 = x'(L) = a \]

Case \( \lambda > 0 \) \( \lambda = \mu^2 \) \[ x'' + \mu^2 x = 0 \] Try \( e^{\mu x} \)
\[ -\mu^2 e^{\mu x} = 0 \Rightarrow \mu = \pm \mu \]

\[ x(x) = A \cos \mu x + B \sin \mu x \]
\[ x'(x) = A \mu \sin \mu x + B \mu \cos \mu x \]

\[ \mu L = \pi \pi \]
\[ \mu = \frac{\pi^2}{L^2} \]
\[ \mu^2 = \left( \frac{\pi^2}{L^2} \right)^2 \]
Eigenvalues \( \lambda = \mu^2 \)
Eigenfunction \( x_n(x) = A_n \cos \frac{n \pi x}{L} \)

Case \( \lambda < 0 \) \( \lambda = -\mu^2 \) \[ x'' - \mu^2 x = 0 \]

Try \( x = e^{\mu x} \)
\[ -\mu^2 e^{\mu x} = 0 \Rightarrow \mu = \pm \mu \]

\[ x(x) = A e^{\mu x} + B e^{-\mu x} \]
\[ = \left( \frac{C_1 + C_2}{2} \right) e^{\mu x} + \left( \frac{C_1 - C_2}{2} \right) e^{-\mu x} \]
\[ = C_1 e^{\mu x} + e^{-\mu x} + C_2 e^{\mu x} - e^{-\mu x} \]
\[ = \frac{C_1 e^{\mu x} + C_2 e^{\mu x}}{2} + C_2 e^{-\mu x} - C_1 e^{-\mu x} \]
\[ = C_1 \cosh \mu x + C_2 \sinh \mu x \]

\[ x'(x) = C_1 \mu \sinh \mu x + C_2 \mu \cosh \mu x \]
\[ 0 = x'(0) = C_2 \mu \Rightarrow C_2 = 0 \]
\[ 0 = x'(L) = C_1 \mu \sinh \mu L \Rightarrow C_1 = 0 \]
Summary when $\lambda = 0$, we have $X_0(x) = \text{const}$

$$X''(x) = 0 \quad \Rightarrow \quad X_1(x) = \frac{(\pi n)^2}{L_0} \quad \text{and} \quad X_N(x) = \cos \frac{n \pi x}{L}$$

$x'(0) = 0 = x'(L)$

Dis Heat conduction

\[ u(x, t) = \text{temperature at position } x \text{ and time } t \]

\[ \alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad 0 < x < L \]

\[ \frac{\partial u}{\partial x} \bigg|_{x=0} = \frac{\partial u}{\partial x} \bigg|_{x=L} = 0 \quad \text{boundary condition} \]

\[ u(x, 0) = f(x) \quad \text{initial conditions} \]