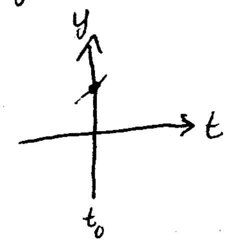
3/5/04

10.1 Two point Boundary Value Problems

$$y'' + p(t)y' + q(t)y = g(t)$$

We've studied <u>initial value problems</u>: $y(t_0) = y_0$
$y'(t_0) = y_1$

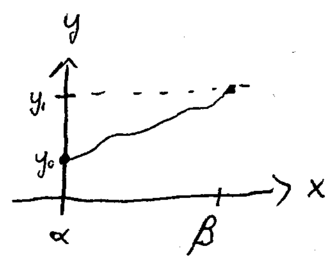
This has a solution.



1) Find general solution
$$y = c_1 y_1(t) + c_2 y_2(t)$$
2) Plug in initial conditions, solve for c_1, c_2. This always works.

Two point boundary value problem

$$\boxed{\begin{gathered} y'' + p(x)y' + q(x)y = g(x) \\ y(\alpha) = y_0 \\ y(\beta) = y_1 \end{gathered}}$$



1) Find general solution.
2) Plug in conditions, but sometimes we can't solve for c_1 and c_2. Sometimes, there are infinitely many solutions.

We need the simplest case:

$$\boxed{\begin{gathered} y'' + \lambda y = 0 \\ y(\alpha) = 0, \quad y(\beta) = 0 \end{gathered}}$$

where λ is a constant.

or $y'(\alpha) = 0$, $y'(\beta) = 0$

homogeneous because of zeros

For some values of λ, this problem has solutions, for other values, it doesn't. The values for which it does have solutions are called <u>eigenvalues</u>.

$$\left(\begin{array}{l} -y'' = \lambda y \\ y(\alpha) = 0 \\ y(\beta) = 0 \end{array} \quad \text{or} \quad \begin{array}{l} -y'' + y' = \lambda y \\ y(\alpha) = 0 \\ y(\beta) = 0 \end{array} \right)$$

This is the operator whose eigenvalues we find

parenthetical remark

$\boxed{\text{Ex}}$ $$X'' + \lambda X = 0$$
$$X(0) = 0 = X(L)$$

<u>Case $\lambda = 0$</u>

Try $X = e^{rx} \Rightarrow r^2 = 0 \Rightarrow r = 0$
$X = b + ax$

$X'' = 0$ general solution is $X = ax + b$

$$0 = X(0) = b$$
$$0 = X(L) = aL + \cancel{b}^{\,0}$$

only solution is $X(x) \equiv 0$. (i.e. $X(x) = 0 \ \forall x$)
This is the trivial solution since there are no nontrivial solutions.

<u>Case $\lambda > 0$</u> $\qquad \lambda = \mu^2 \qquad X'' + \mu^2 X = 0$

Try $X = e^{rx}$ plug in $r^2 + \mu^2 = 0$
$r = \pm \mu i$

$$X(x) = \cancel{A}^{\,0} \cos \mu x + B \sin \mu x$$
$$0 = X(0) = A$$
$$0 = X(L) = B \sin \mu L \Rightarrow \mu L = n\pi$$

$\longrightarrow$

$\mu = \frac{n\pi}{L}$ $\qquad \lambda = \mu^2 = \left(\frac{n\pi}{L}\right)^2$

Corresponding solutions are

$X(x) = B \sin \frac{n\pi x}{L}$ ← eigenfunction corresponding to eigenvalue

$n=1$

$\lambda = \left(\frac{\pi}{L}\right)^2$

$X_1(x) = \sin \frac{\pi x}{L}$

$n=2$:

$\lambda = \left(\frac{2\pi}{L}\right)^2$

$X_2(x) = \sin \frac{2\pi x}{L}$

<u>Case $\lambda < 0$</u> $\qquad \lambda = -\mu^2 \qquad X'' - \mu^2 X = 0$

Try $X = e^{rx}$ $\quad r^2 - \mu^2 = 0 \Rightarrow r = \pm \mu$

general solution

$$X(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

Change basis

$$= k_1 \underbrace{\left(\frac{e^{\mu x} + e^{-\mu x}}{2}\right)}_{\cosh \mu x} + k_2 \underbrace{\left(\frac{e^{\mu x} - e^{-\mu x}}{2}\right)}_{\sinh \mu x}$$

$$= e^{\mu x} \underbrace{\left(\frac{k_1 + k_2}{2}\right)}_{c_1} + e^{-\mu x} \underbrace{\left(\frac{k_1 - k_2}{2}\right)}_{c_2}$$

This shows how the constants c_1, c_2 are related to the constants k_1, k_2.

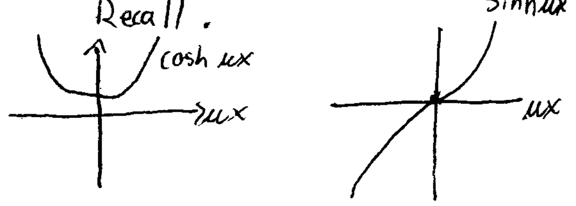
General solution is $X(x) = k_1 \cosh(\mu x) + k_2 \sinh(\mu x)$

$$X(x) = \overset{0}{\cancel{k_1}} \cosh \mu x + k_2 \sinh \mu x$$

$$0 = X(0) = k_1$$

$$0 = X(L) = k_2 \underbrace{\sinh \mu L}_{\text{zero only if } \mu = 0}$$

Recall:



We only get the trivial solution $(X \equiv 0)$.

zero only if $\mu = 0$
Not allowed for $\lambda < 0$.

For $X'' + \lambda X = 0$

$X(0) = 0 = X(L)$

We have nonzero/nontrivial solutions only when $\lambda = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2 \ldots$

The corresponding solutions are $X(x) = \sin \frac{n\pi x}{L}$