1. Note first:

$$\exists\, x > 0 \text{ with } Ax = 0 \iff \exists\, x \geq e \text{ with } Ax = 0,$$

where e = vector of all ones.

Since: $\Leftarrow$: trivial

$\Rightarrow$: rescale x so that it is $\geq e$.

Consider the LP pair:

$$\begin{array}{llll} \min & 0 & & \\ \text{s.t.} & Ax = 0 & & (P) \\ & x \geq e & & \end{array} \qquad \begin{array}{llll} \max & e^T s & & \\ \text{s.t.} & A^T y + s = 0 & & (D) \\ & s \geq 0 & & \end{array}$$

Assume $\exists\, x > 0$ with $Ax = 0$:

Then (P) is feasible, with optimal value 0

So (D) has optimal value 0

So $A^T y \leq 0 \Rightarrow A^T y = 0$.

So $\nexists\, y$ with $A^T y \leq 0$, $A^T y \neq 0$.

Assume $\nexists\, x > 0$ with $Ax = 0$:

Then (P) is infeasible

(D) is always feasible take $y = 0, s = 0$.

So (D) is unbounded.

So $\exists\, y$ with $A^T y \leq 0$, $A^T y + s = 0$, $s \neq 0$

So $\exists\, y$ with $A^T y \leq 0$, $A^T y \neq 0$.

2. (a) $A = [1 \quad -1 \quad 3]$, $\quad c^T = [2 \quad -3 \quad 4]$

$c^T x^1 = 2, \quad c^T x^2 = 4, \quad A x^1 = 1, \quad A x^2 = 3$

Initial (RMP):

$$\begin{aligned} \min \quad & 2\lambda_1 + 4\lambda_2 \\ \text{s.t.} \quad & \lambda_1 + 3\lambda_2 = 2 \\ & \lambda_1 + \lambda_2 = 1 \\ & \lambda_i \geq 0 \end{aligned} \qquad \text{(RMP)}$$

Only feasible solution: $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{2}$, value 3.

Dual solution: $\pi = 1$, $\sigma = 1$, so $B^T \binom{\pi}{\sigma} = c$.

Subproblem: $c^{\cancel{T}} - A^T \pi = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$\begin{aligned} \min \quad & x_1 - 2x_2 + x_3 \\ \text{s.t.} \quad & x \in X. \end{aligned} \qquad \text{(SP)}$$

Unbounded: Soln gives ray $d^1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

$c^T d^1 = -3, \quad A d^1 = -1$

(RMP) becomes:

$$\begin{aligned} \min \quad & 2\lambda_1 + 4\lambda_2 \cancel{+} -3\mu_1 \\ \text{s.t.} \quad & \lambda_1 + 3\lambda_2 - \mu_1 = 2 \\ & \lambda_1 + \lambda_2 = 1 \\ & \lambda_i \geq 0 \end{aligned} \qquad \text{(RMP)}$$

Optimal soln: $\lambda_2 = 1$, $\mu_1 = 1$, $B = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$, value $= 1$.

Dual solution: $\pi = 3$, $\sigma = -5$

Subproblem: $c - A^T\pi = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} - 3\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -5 \end{bmatrix}$

$$\begin{aligned} \min \quad & -x_1 - 5x_2 \\ \text{st.} \quad & x \in X \end{aligned}$$

Soln: $x^3 = (1, 0, 1)^T$, value $= -6 < \sigma$.

$c^T x^3 = 6$, $\quad Ax^3 = 4$

(RMP) becomes:

$$\begin{aligned} \min \quad & 2\lambda_1 + 4\lambda_2 + 6\lambda_3 - 3\mu_1 \\ \text{st.} \quad & \lambda_1 + 3\lambda_2 + 4\lambda_3 - \mu_1 = 2 \qquad \text{(RMP)} \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_i \geq 0 \end{aligned}$$

Optimal soln: $\lambda_3 = 1$, $\mu_1 = 2$, value $= 0$, $B = \begin{bmatrix} 4 & -1 \\ 1 & 0 \end{bmatrix}$

Dual solution: $\pi = 3$, $\sigma = -6$

Since π is unchanged, the optimal soln to the subproblem is x^3, with value $= \sigma = 6$

So OPTIMAL: $\boxed{x = x^3 + 2d^1 = (1, 2, 1), \text{ value} = 6\lambda_3 - 3\mu_1 = 0}$

(b) $\min \; 2x_1 - 3x_2 + 4x_3$

s.t. $x_1 - x_2 + 3x_3 = 2$

$0 \le x_1 \le 1, \; 0 \le x_2, \; 0 \le x_3 \le 1.$

Initialize with x_3 basic, x_1 nonbasic at lower bound, x_2 nonbasic.

Pivot: $\min \; \frac{2}{3}x_1 - \frac{5}{3}x_2 + 8/3$ BFS: Value: $8/3$

s.t. $\frac{1}{3}x_1 - \frac{1}{3}x_2 + x_3 = 2/3$ $x_1 = 0, \; x_2 = 0, \; x_3 = 2/3$

$0 \le x_1 \le 1, \; 0 \le x_2, \; 0 \le x_3 \le 1$

x_2 enters.

This does not give a ray, because x_3 reaches its upper bound.

$\min \; -x_1 - 5x_3 + 6$ BFS: Value: $-5+6=1$

s.t. $-x_1 + x_2 - 3x_3 = -2$ $x_1 = 0, \; x_2 = 1, \; x_3 = 1$

$0 \le x_1 \le 1 \quad 0 \le x_2 \quad 0 \le x_3 \le 1$ LB B UB

x_1 enters.

This does not give a ray, because x_1 then leaves at its upper bound

Get same tableau ~~pexcept~~:

$\min \; -x_1 - 5x_3 + 6$ BFS: Value $= -1-5+6=0$

s.t. $-x_1 + x_2 - 3x_3 = -2$ $x_1 = 1, \; x_2 = 2, \; x_3 = 1$

$0 \le x_1 \le 1, \; 0 \le x_2, \; 0 \le x_3 \le 1$ UB B UB

OPTIMAL: $\boxed{\text{Value} = 0 \quad x_1 = 1, \; x_2 = 2, \; x_3 = 1}$

3. (a) Dual problem to (MP) is:

$$\begin{aligned} \max \quad & e^T y + K\sigma \\ \text{s.t.} \quad & A^T y + e\sigma \le w \end{aligned} \qquad (MD).$$

Solving the (RMP) gives a dual solution, $(\bar{y}, \bar{\sigma})$, using complementary slackness. If $(\bar{y}, \bar{\sigma})$ is dual feasible in (MD), then we have the optimal solution.

Thus, we need to check:

Is $(a^k)^T \bar{y} + \sigma \le w_k$ for every subset S^k?

(Here, a^k is the incidence vector of S^k)

To determine this, we need to solve the subproblem:

$$\begin{aligned} \min \quad & w - a^T \bar{y} \\ \text{s.t.} \quad & a \text{ is binary, } w \text{ is weight of } a. \end{aligned} \qquad (SP)$$

If optimal value is $\ge \sigma$ then we have solved (MP); otherwise, we find a subset (or column) with negative reduced cost.

Note that $w = \sum_{i,j \in S^k,\, i<j} d_{ij} = \sum_{i<j} d_{ij} a_i a_j$,

where a_i and a_j are the ith & jth component of a.

Thus, the subproblem can be written:

$$\min \sum_{i<j} d_{ij} a_i a_j - \bar{y}^T a \qquad (QP)$$

$$\text{s.t.} \quad a \text{ is binary.}$$

This is a quadratic program, with integrality constraints.

(b) (i) (RMP) is:

$$
\begin{array}{llr}
\min & 8x_{12} + 2x_{34} & \\
\text{s.t.} & x_1 + x_{12} = 1 & (1) \\
& x_2 + x_{12} = 1 & (2) \\
& x_3 + x_{34} = 1 & (3) \\
& x_4 + x_{34} = 1 & (4) \\
& x_5 = 1 & (5) \\
& x_1 + x_2 + x_3 + x_4 + x_5 + x_{12} + x_{34} = 3 & (6) \\
& x \geq 0 &
\end{array}
$$

(5) $\Rightarrow x_5 = 1$. (1), (2) $\Rightarrow x_1 = x_2 = 1 - x_{12}$. (3), (4) $\Rightarrow x_3 = x_4 = 1 - x_{34}$.

Substituting into (6) gives: $5 - x_{12} - x_{34} = 3$

(2) & (3) $\Rightarrow x_{12} \leq 1$, $x_{34} \leq 1$.

So we must have $x_{12} = 1$, $x_{34} = 1$, $x_5 = 1$, $x_1 = x_2 = x_3 = x_4 = 0$

(ii) Put $x_1, x_2, x_3, x_5, x_{12}, x_{34}$ in the basis, so

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Need to check the reduced costs, ie, need to check dual feasibility.

Dual solution ~~is $(\bar{y}, \bar{\sigma}) = B$~~ satisfies:

$$B^T \begin{pmatrix} \bar{y} \\ \bar{\sigma} \end{pmatrix} = c_B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 8 \\ 2 \end{pmatrix}$$

ie,

$$\begin{pmatrix} 1 & & & & & 1 \\ & 1 & & & & 1 \\ & & 1 & & & 1 \\ & & & & 1 & 1 \\ 1 & 1 & & & & 1 \\ & & 1 & 1 & & 1 \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \\ \bar{y}_4 \\ \bar{y}_5 \\ \bar{\sigma} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 8 \\ 2 \end{pmatrix}$$

Solution is: $\bar{\sigma} = -8$, $\bar{y}^T = (8, 8, 8, 2, 8)$

Reduced cost for x_4 is: $c_4 - c_B^T B^{-1} \begin{pmatrix} a_4 \\ 1 \end{pmatrix}$

ie, $0 - \bar{y}^T a_4 - \bar{\sigma} = -2 + 8 = 6 \geq 0$.

So all the reduced costs are nonnegative.

(iii) From part (a), the subproblem is:

$$\min \sum_{i<j} d_{ij} a_i a_j - \bar{y}^T a \qquad (QP).$$

$$\text{s.t.} \quad a \text{ is binary}$$

The vector $a = (0, 0, 1, 1, 1)$ has value $9 - 18 = -9 < -8 = \bar{\sigma}$.

So we are not optimal for (MP).

The column to add to (RMP) is: $\begin{pmatrix} a \\ 1 \end{pmatrix}$. Note that $w_{345} = 9$.

We need to find the corresponding column in the tableau,

given by $B d = \begin{pmatrix} a \\ 1 \end{pmatrix}$

Solve:
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \\ \\ d \\ \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix};$$

Solution is: $d^T = (-1, -1, 0, 1, 1, 1)$

To determine the leaving variable, we need to conduct a min ratio test:

Compare d with $B^{-1}b$

$$\begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{matrix}$$

So x_{345} comes in with value $\Delta = 1$, and any one of x_5, x_{12}, x_{34} can leave the basis. Let x_{12} leave the basis

Now have

and $x_{345} = 1$.

$$x = \cancel{\begin{pmatrix} x_1 \\ x_{345} \\ x_3 \\ x_5 \\ x_{34} \\ x_{345} \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_5 \\ \cancel{x_{345}}\ x_{12} \\ x_{34} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \Delta d = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So we now have $x_1 = 1$, $x_2 = 1$, $x_{345} = 1$, $\underbrace{x_3 = x_5 = x_{34} = 0}_{\text{basic}}$,

$\underbrace{x_4 = x_{12}}_{\text{nonbasic.}} = 0$ Basic variables are

$x_1, x_2, x_3, x_5, x_{345}, x_{34}$, in order.

Need to find new y:, solve $B^T y = c_B$.

This system is now:

$$\begin{pmatrix} 1 & & & & & 1 \\ & 1 & & & & 1 \\ & & 1 & & & 1 \\ & & & & 1 & 1 \\ & & 1 & 1 & 1 & 1 \\ & & 1 & 1 & & 1 \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \\ \bar{y}_4 \\ \bar{y}_5 \\ \bar{\sigma} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 9 \\ 2 \end{pmatrix}$$

Solution is: $\bar{y}^T = (7, 7, 7, 2, 7)$, $\bar{\sigma} = -7$.

Need to check dual feasibility to determine if optimal:

Reduced costs $= c_N^T - c_B^T B^{-1} N = c_N^T - (\bar{y}^T \ \bar{\sigma}) N$

$$= (\underset{\text{set } \{4\}}{0}, \underset{\text{set } \{1,2\}}{8}) - (7\ 7\ 7\ 2\ 7\ -7) \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= (0, 8) - (-5, 7) = (5, 1).$$

Since reduced costs are nonnegative, YES, we do solve (RMP).

4. (a) $x = e$, $t = 1$, $w = 1$, $y = 0$.

(b) Dual to (HLP) is:

$$
\begin{array}{lrcl}
\max & & -(n+1)\zeta & \\
\text{s.t.} & -A\nu + b\rho - \hat{b}\zeta & = & 0 \\
& A^T\pi - c\rho + \hat{\partial}\zeta & \leq & 0 \\
& -b^T\pi + c^T\nu - \hat{\partial}\zeta & \leq & 0 \\
& \hat{b}^T\pi - \hat{c}^T\nu + \hat{\partial}\rho & = & 0 \\
& \nu, \rho & \geq & 0.
\end{array}
\qquad (HLD)
$$

Change sign of objective, change sign of constraints:

get exactly (HLP) with $\pi \leftrightarrow x$, $\nu \leftrightarrow y$, $\rho \leftrightarrow t$, $\zeta \leftrightarrow w$.

(c) Optimal value of (HLP) = optimal value of (HLD)

If (x^*, y^*, t^*, w^*) solves (HLP), then it

also solves (HLD).

So $(n+1)w^* = -(n+1)w^*$

So $w^* = 0$, and optimal value $= 0$.

(d) Let $x^* = \tilde{x}/\tilde{t}$, $y^* = \tilde{y}/\tilde{t}$.

Since $\tilde{w} = 0$, we get $Ax^* = b$, $A^T y^* \leq c$,
$b^T y^* = c^T x^*$, $x^* \geq 0$, $y^* \geq 0$.

(e) If $t = 0$ and $w = 0$:

Then
$$A x = 0 \quad (1)$$
$$A^T y \leq 0 \quad (2)$$
$$b^T y \geq c^T x \quad (3)$$

Further, t is complementary to the slack variable in (3).

So, in a strictly complementary soln, if $t = 0$ then

$$b^T y > c^T x \quad (4).$$

If $c^T x \geq 0$: Then $A^T y \leq 0$ and $b^T y > 0$.
So by Farkas, $\nexists\, x$ with $Ax = b$, $x \geq 0$.

If $c^T x < 0$. ~~The~~ Assume $\bar{y}$ satisfies $A^T \bar{y} \leq c$.
Then $x^T A^T \bar{y} \leq c^T x$ since $x \geq 0$.
But this gives $0^T \bar{y} \leq c^T x < 0$ ⨳.

So: either $Ax = 0$ and $c^T x < 0$, which implies (D) is infeasible
or $A^T y \leq 0$ and $b^T y > 0$, which implies (P) is infeasible.
Note: may have $Ax = 0$, $c^T x < 0$ AND $A^T y \leq 0$ $b^T y > 0$.