Differential Equation

An equation containing derivatives is called a differential equation.

Notation: D.E.

a) An equation involving only ordinary derivatives is called an ordinary differential equation. Notation: O.D.E.

b) An equation involving partial derivatives is called a partial differential equation. Notation: P.D.E.

c) We shall consider an equation having the form:

\[ \frac{dX}{dt} = AX + G, \]

where \( X \) is a vector, \( A \) is a given matrix, and \( G \) is a vector valued function. This type of equation is called a system of differential equations.

Order

The highest order of derivation is called order of the D.E.

Examples:

a) \( y'' + y = 0 \) is a first order O.D.E.

b) \( x \frac{du}{dx} + \frac{du}{dy} = 0 \) is a first order P.D.E.

c) \( \frac{3u}{3t} = \frac{3u}{3t} \) is a second order P.D.E.

d) \( y''' - y = 0 \) is a third order O.D.E.
\[ \begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= -x
\end{align*} \]

is a system of ODEs \( \text{of order one} \)

3. **Solution**

Suppose that \( f \) is continuously differentiable up to order \( n \) on an interval \( I \), and satisfies an ODE \( (E) \) of order \( n \), then \( f \) is called a solution of \( E \).

**Examples**

a) \( y'' + y = 0 \) \( \text{(E)} \)

Clearly, for each constant \( c \),

\[ f(x) = c \cos x, \quad \infty < x < \infty \]

is a solution of \( (E) \)

b) \( y'' + w^2 y = 0 \)

Since \( (\cos wx)' = -w \sin wx, \cos wx \) is a solution.

Similarly, \( \sin wx \) is also a solution.

**General solution**

A formula describing all solutions of a D.E. is called the general solution of the equation.

**Particular solution**

A solution of a D.E., which is free from arbitrary constants, is called a particular solution.
Example: \( y' + y = 0 \)

Clearly, the constant 1 satisfies this D.E. It is a particular solution. But all solutions have the form: \( y(x) = Ae^{-x} \), where \( a \) is an arbitrary constant. Therefore \( y_p(x) = 1 \) is a particular solution and \(Ce^{-x}+1 \) is the general solution.

4) Linearity

An \( n \)th order O.D.E. having the form:
\[
\gamma^{(n)} + a_1(x)\gamma^{(n-1)} + \ldots + a_n(x)\gamma = g(x), \quad x \in I
\]
where \( a_1, \ldots, a_n \) and \( g \) are functions of \( x \) only, is called linear. Otherwise, the equation is said to be nonlinear.

Examples:

a) \( y'' + x^{-1}y' + y = 0 \), \( x > 0 \), is linear

b) \( y'' + k\sin y = 0 \), is nonlinear

5) Initial and boundary conditions

Consider the linear D.E.
\[
y'' + a_1(x)y' + a_2(x)y = g(x), \quad x \in I
\]
and suppose that \( y(x), y'(x) \) are given at some point \( x_0 \) in I. The conditions of \( y \) and its derivative at \( x_0 \) are called initial conditions. The problem of solving the D.E. subject to the initial conditions is an initial value problem.

Notation: I.V.P.

Consider the linear D.E.
\[
y'' + a_1(x)y' + a_2(x)y = 0, \quad a \in [0, b]
\]
with the boundary conditions:
\[
\begin{align*}
\alpha_1 y(a) + \alpha_2 y'(a) &= 0 \\
\beta_1 y(b) + \beta_2 y'(b) &= 0
\end{align*}
\]
The problem of solving the D.E. subject to the boundary conditions is called a boundary value problem.

Notation: B.V.P.

Examples:

1) Consider the D.E.

\[ y'' = -g \quad \text{where} \quad \frac{d}{dt} \]

and \( g \) is the constant of gravity.

By integrating twice, we get the general solution

\[ y(t) = -\frac{1}{2} gt^2 + c_1 t + c_2 \]

where \( c_1 \) and \( c_2 \) are constants of integration.

Suppose we are given the initial position and the initial velocity: \( y(0) = y_0 \) and \( y'(0) = v_0 \)

which are the initial conditions, then it is possible to obtain the unique solution of this I.V.P., which is simply

\[ y(t) = -\frac{1}{2} gt^2 + v_0 t + y_0 \]

2) Consider the B.V.P.

\[ y'' + ky = 0 \quad , \quad 0 < x < 1 \]

with the B.C.: \( y(0) = 0 = y(1) \)

The general solution is

\[ y(x) = a \cos kx + b \sin kx \]

where \( a \) and \( b \) are arbitrary constants.

 Applying the B.C., we get:

\[ 0 = y(0) = a \]

\[ 0 = y(1) \Rightarrow \sin k = 0 \Rightarrow k = n\pi \]

In order to have nontrivial solutions, we must have \( k^2 = (n\pi)^2 \), \( n = 0, 1, 2, \ldots \), and each \( y_n(x) = \sin n\pi x \) is a solution.