Subspaces of $\mathbb{R}^n$

**Problem 1.** Let $S$ be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad v = \begin{bmatrix} 4 \\ 2 \\ 1 \\ 5 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 7 \end{bmatrix}.$$ 

Determine if the vectors $x = \begin{bmatrix} 8 \\ 9 \\ 5 \\ 16 \end{bmatrix}$ and $y = \begin{bmatrix} 7 \\ 2 \\ 1 \\ 3 \end{bmatrix}$ are in $S$.

**Solution by Maple.** Enter and name the five vectors

> $u := \text{vector}([1, 2, 3, 4]); v := \text{vector}([4, 2, 1, 5]); w := \text{vector}([3, 5, 1, 7]);$
> $x := \text{vector}([8, 9, 5, 16]); y := \text{vector}([7, 2, 1, 3]);$

We need to determine if $x$ and $y$ are linear combinations of the vectors $u, v$ and $w$.

We can answer both questions by row-reducing the matrix

> $A := \text{augment}(u, v, w, x, y);$  
> $A := \begin{bmatrix} 1 & 4 & 3 & 8 & 7 \\ 2 & 2 & 5 & 9 & 2 \\ 3 & 1 & 1 & 5 & 1 \\ 4 & 5 & 7 & 16 & 3 \end{bmatrix}$

> $\text{rref}(A);$  
> $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

We see that $x$ is in $S$, but $y$ is not in $S$.

(over)
**Problem 2.** Determine if the five vectors span \( \mathbb{R}^4 \):

\[
\begin{bmatrix}
2 \\
1 \\
3 \\
5
\end{bmatrix}, \quad
\begin{bmatrix}
4 \\
3 \\
7 \\
2
\end{bmatrix}, \quad
\begin{bmatrix}
3 \\
1 \\
2 \\
5
\end{bmatrix}, \quad
\begin{bmatrix}
2 \\
1 \\
2 \\
5
\end{bmatrix}, \quad
\begin{bmatrix}
4 \\
1 \\
3 \\
1
\end{bmatrix}.
\]

**Solution by Maple.** The vectors span \( \mathbb{R}^4 \) if the system

\[
\begin{bmatrix}
2 & 4 & 3 & 2 & 4 \\
1 & 3 & 1 & 1 & 1 \\
3 & 7 & 2 & 5 & 3 \\
5 & 2 & 5 & 6 & 1
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
c_5
\end{bmatrix}
=
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{bmatrix}
\]

has a nontrivial solution for every vector \( v \) in \( \mathbb{R}^4 \), that is, \( v \) is a linear combination of the vectors. Construct the coefficient matrix of the system and row-reduce it.

\[
> A := \text{augment}(u1, u2, u3, u4, u5);
\]

\[
A :=
\begin{bmatrix}
2 & 4 & 3 & 2 & 4 \\
1 & 3 & 1 & 1 & 1 \\
3 & 7 & 2 & 5 & 3 \\
5 & 2 & 5 & 6 & 1
\end{bmatrix}
\]

\[
> \text{rref}(A);
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & -\frac{57}{11} \\
0 & 1 & 0 & 0 & \frac{5}{11} \\
0 & 0 & 1 & 0 & \frac{32}{11} \\
0 & 0 & 0 & 1 & \frac{21}{11}
\end{bmatrix}
\]

The reduced form of the coefficient matrix has a 1 in each row. Thus the system has a nontrivial solution for every \( v \). We can conclude that the five vectors span \( \mathbb{R}^4 \). \textit{Note:} Call \( W = \{u1, u2, u3, u4, u5\} \). Since \( W \) has 5 vectors, each of dimension 4, by Theorem 11 of chapter 1, \( W \) is linearly dependent. Thus \( W \) cannot be a \textit{basis} for \( \mathbb{R}^4 \). (See Definition 4 of chapter 3.)