Name:

MATP6620/DSES6770

Combinatorial Optimization and Integer Programming

Spring 2001

Midterm Exam, Thursday, March 8, 2001.

Please do all five problems. Show all work. No books or calculators allowed. You may use any result from class, the homeworks, or the texts, except where stated. You may use one sheet of handwritten notes. The exam lasts two hours.

SOLUTIONS.

Q1	
Q2	
Q3	
Q4	
Q5	
Total	

1. (20 points) An instance d of a feasibility problem $X \in NP$ depends upon two positive integer parameters m and n. Assume d requires storage $m2^n$ in binary, and that we know an algorithm A which solves d in time 2^{m+n}.

   (a) (~~8~~ 10 points) Can we conclude X is in P?

   ~~(b) (6 points) Can we conclude X is not in P?~~

   (~~c~~ b) (~~8~~ 10 points) Assume we know in addition that $m \le n$ for every instance d of X. What can we conclude now?

(a) No: E.g., if n is fixed, the runtime is

$$2^{m+n} = c\,2^{\text{storage}}.$$

(b) ~~No:~~ ~~E.g.,~~ If m is bounded in terms of n, then runtime is:

$$2^{m+n} \le 2^{2n} = \frac{(m2^n)^2}{m^2}$$

$$\le c\,(m2^n)^2,$$ for some positive fixed c, since m is positive,

So it is bounded by a polynomial in the storage requirement.

2. (20 points) Using the *Hamiltonian path* problem, or otherwise, show that the following problem is $\mathcal{NP}$-complete.

   Given a graph $G = (V, E)$ and an integer k, is there a spanning tree T of G that has exactly k leaves?

   (A *leaf* of a tree is a vertex of degree 1. The Hamiltonian path problem is: Given a graph $G = (V, E)$, does there exist a path which visits all the vertices of G exactly once? You may assume that this problem is $\mathcal{NP}$-complete.)

We reduce HAMILTONIAN PATH to the leaves problem:

Given an instance of Hamiltonian path:

Keep the same graph.

Set $k=2$.

Is there a spanning tree with 2 leaves?

Note that any Hamiltonian path is a spanning tree with two leaves. Conversely, any spanning tree with two leaves must be a Hamiltonian path, since the spanning tree has $|V|-1$ edges, so the sum of the degrees of the nodes is $2(|V|-1)$. There are two leaves, so the sum of the degrees of the remaining $|V|-2$ vertices is $2(|V|-2)$. Since they all have degree at least 2, they must have degree exactly two, so we have a Hamiltonian path.

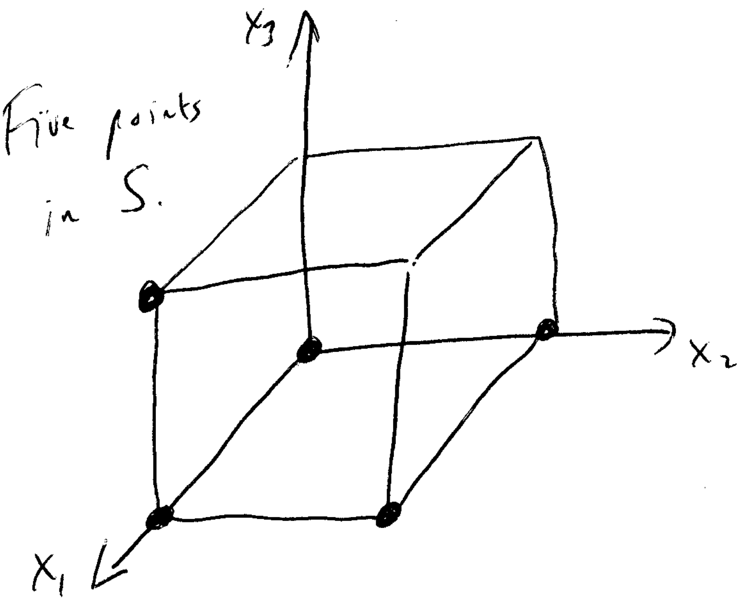
3. Let $S := \{x \in \mathbb{B}^3 : -x_1 + x_2 + 3x_3 \leq 2\}$. Let $S^0 := S \cap \{x \in \mathbb{B}^3 : x_1 = 0\}$.

   (a) (5 points) Show that $x_2 \leq 1$ defines a facet of the convex hull of S^0.

   (b) (5 points) Derive a valid inequality for S by lifting the valid inequality for S^0 given in part 3a.

   (c) (5 points) Show that the inequality you derived in part 3b does not define a facet of the convex hull of S.

   (d) (5 points) Give an inequality description of the convex hull of S.



(a) S^0 consists of the two points $(0,1,0)$ and $(0,0,0)$. So $\text{conv}(S^0)$ is the line segment between these two points, and the points are the facets. $x_2 \leq 1$ defines the facet $(0,1,0)$.

(b) Set $x_1 = 1$. Solve

$$\begin{aligned} \max \quad & x_2 \\ \text{s.t.} \quad & -1 + x_2 + 3x_3 \leq 2 \\ & x \text{ binary.} \end{aligned}$$

Solution has value 1, achieved at the point $(1,1,0)$.

So coefficient of x_1 is $(1-1) = 0$.

Lifted constraint: $x_2 \leq 1$.

(c) S is full dimensional: it contains the origin and the three linearly independent points $(1,0,0)$, $(0,1,0)$, $(1,0,1)$. But the face $x_2 = 1$ only contains two points from S, namely $(0,1,0)$ and $(1,1,0)$, so it has dimension 1, while the facets have dimension 2.

(d)

$$x_1 \qquad - x_3 \geq 0$$

(goes through $(1,0,1)$, $(0,0,0)$, $(1,1,0)$)

$$x_2 + x_3 \leq 1$$

(goes through $(0,1,0)$, $(1,1,0)$, $(1,0,1)$)

$$x_1 \leq 1$$

$$x_3 \geq 0$$

$$x_2 \geq 0$$

This is a minimal description.
Each of these is a facet, going through at least three of the points in S.

4. (20 points) We wish to solve the MAXCUT problem on a graph $G = (V, E)$ with edge weights c_e. Recall that we modeled this as an integer programming problem by introducing binary variables x_e to indicate whether an edge is in the cut. Let S be the set of incidence vectors corresponding to cuts. These variables satisfied the constraints

$$x(F) - x(C \setminus F) \leq |F| - 1 \tag{1}$$

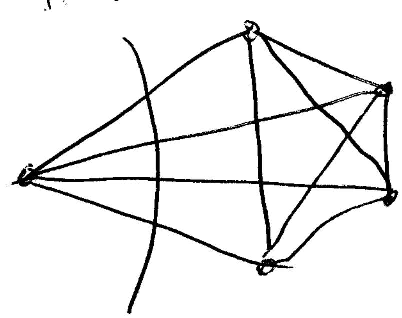
for any chordless cycle C, where F is a subset of C of odd cardinality. We are going to restrict our attention to K_5, the complete graph on 5 vertices.

(a) (5 points) Argue from first principles that any maxcut on K_5 uses at most 6 of the edges of K_5.

(b) (10 points) Show that the point $x_e = \frac{2}{3}$ for all edges e satisfies (1) for all chordless cycles C and corresponding subsets F. Show further that this point is not in the convex hull of the set of incidence vectors of cuts. What constraint does this suggest?

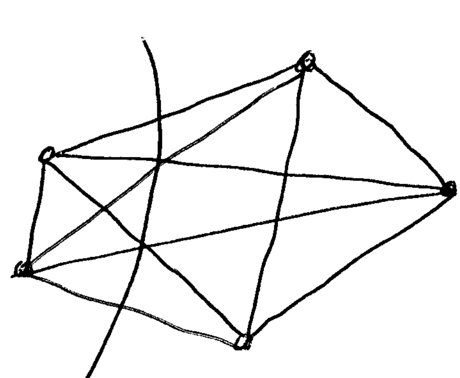
(c) (5 points) How would you try to show that the constraint you defined in part 4b gives a facet of the convex hull of cuts? (Note: I do not want you to show that it is a facet; instead, I want you to tell me what points you might consider, and what you might try to do with those points. You may assume S is full dimensional.)

(a) If one vertex is on one side, and four are on the other side:



Four edges are cut.

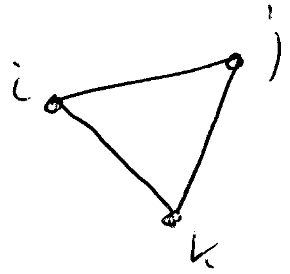
If two vertices are on one side, and three on the other:



Six edges are cut.

So at most six edges appear in any cutset.

(b) The chordless cycles are all of length 3.



If all three edges are in F:

$$x_{ij} + x_{jk} + x_{ik} \leq 2.$$

Satisfied by $x_e = \frac{2}{3}$ $\forall e$.

If only one edge in F:

$$x_{ij} - x_{jk} - x_{ik} \leq 0$$

Satisfied by $x_e = \frac{2}{3}$ $\forall e$.

So $x_e = \frac{2}{3}$ $\forall e$ satisfies the constraints.

Now, $\sum x_e = 10\left(\frac{2}{3}\right)$ since K_5 has 10 edges

$= \frac{20}{3} = 6\frac{2}{3}$.

But, from part (a), any ~~vertex~~ point in the convex hull has $\sum x_e \leq 6$

Thus, we could add the constraint:

$$\sum_{e \in E} x_e \leq 6.$$

(c) We have 10 edges, so the dimension of the convex hull is 10.

Thus, we need to find 10 affinely indep points.

There are 10 cutsets that satisfy the constraint at equality:

Place two vertices on one side, three on the other.

Cutsets: $\{12\}$ $\{13\}$ $\{14\}$ $\{15\}$ $\{23\}$ $\{24\}$ $\{25\}$ $\{34\}$ $\{35\}$ $\{45\}$

Edges:

$$\begin{array}{c} 12 \\ 13 \\ 14 \\ 15 \\ 23 \\ 24 \\ 25 \\ 34 \\ 35 \\ 45 \end{array} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{etc.}$$

So we need to show that these ten vectors are affinely independent.

5. (20 points) *Resolution* is a technique used in logic to try to solve instances of SATISFIABILITY. It works by combining two clauses to generate an extra clause that must be satisfied by any solution to the instance. For example, consider the two clauses $x_1 \vee \bar{x}_2 \vee x_3$ and $\bar{x}_2 \vee \bar{x}_3 \vee x_4$. If x_3 takes the value true then either x_2 must be false or x_4 must be true in order for the second clause to be satisfied. If x_3 is false then either x_1 must be true or x_2 must be false in order for the first clause to be satisfied. It follows that any truth assignment satisfying the clauses $x_1 \vee \bar{x}_2 \vee x_3$ and $\bar{x}_2 \vee \bar{x}_3 \vee x_4$ must also satisfy the clause $x_1 \vee \bar{x}_2 \vee x_4$. We say in this case that the two clauses were *resolved using* x_3.

   In general, to resolve two clauses using variable x_i, the clauses have the forms:

$$x_i \quad \vee \quad (\bigvee_{j \in C_1^+} x_j) \vee (\bigvee_{j \in C_1^-} \bar{x}_j) \tag{2}$$

$$\bar{x}_i \quad \vee \quad (\bigvee_{j \in C_2^+} x_j) \vee (\bigvee_{j \in C_2^-} \bar{x}_j). \tag{3}$$

   Any solution to these two clauses must also satisfy the *resolved clause*

$$(\bigvee_{j \in C_1^+ \cup C_2^+} x_j) \vee (\bigvee_{j \in C_1^- \cup C_2^-} \bar{x}_j). \tag{4}$$

   Note that it is not necessary that the four sets of indices C_1^+, C_1^-, C_2^+, and C_2^- be distinct.

   In this question, we are going to investigate the representation of SATISFIABILITY as an integer programming problem. Thus, we introduce a 0-1 variable y_i for each logical variable x_i and we introduce an inequality for each clause. The clauses can be satisfied simultaneously if and only if the inequalities can be satisfied simultaneously.

   (a) (5 points) Assume the four sets C_1^+, C_1^-, C_2^+, and C_2^- are distinct. Show that the inequality corresponding to (4) is implied by the inequalities corresponding to equations (2) and (3).

   (b) (10 points) Now assume that $C_i^+ \cap C_j^- = \emptyset$ for any combination of i and j, but that either $C_1^+ \cap C_2^+ \neq \emptyset$ and/or $C_1^- \cap C_2^- \neq \emptyset$. Show that the resolved clause can be derived using one step of the Chvatal-Gomory rounding procedure.

   (c) (5 points) Find a fractional point that satisfies the inequalities corresponding to the clauses $x_1 \vee \bar{x}_2 \vee x_3$ and $\bar{x}_2 \vee \bar{x}_3 \vee x_4$ but which violates the inequality corresponding to the clause $x_1 \vee \bar{x}_2 \vee x_4$.

(a) Ineqs corresponding to (2) and (3) are:

$$y_i + \sum_{j \in C_1^+} y_j + \sum_{j \in C_1^-} (1-y_j) \geq 1 \qquad (2')$$

$$1-y_i + \sum_{j \in C_2^+} y_j + \sum_{j \in C_2^-} (1-y_j) \geq 1 \qquad (3')$$

Add:

$$1 + \sum_{j \in C_1^+} y_j + \sum_{j \in C_2^+} y_j + \sum_{j \in C_1^-} (1-y_j) + \sum_{j \in C_2^-} (1-y_j) \geq 2 \quad (5)$$

Rearrange:

$$\sum_{j \in C_1^+ \cup C_2^+} y_j + \sum_{j \in C_1^- \cup C_2^-} (1-y_j) \geq 1$$

This is the inequality corresponding to (4).

(b) Rearrange (5) under the assumptions of this part:

$$2 \sum_{j \in C_1^+ \cap C_2^+} y_j + \sum_{j \in C_1^+ \setminus C_2^+} y_j + \sum_{j \in C_2^+ \setminus C_1^+} y_j$$

$$+ 2 \sum_{j \in C_1^- \cap C_2^-} (1-y_j) + \sum_{j \in C_1^- \setminus C_2^-} (1-y_j) + \sum_{j \in C_2^- \setminus C_1^-} (1-y_j) \geq 1$$

Divide by 2 and round UP (since we have $\geq$ constraint):

$$\sum_{j \in C_1^+ \cup C_2^+} y_j + \sum_{j \in C_1^- \cup C_2^-} (1-y_j) \geq 1$$

This is the required inequality.

(c) $\boxed{y_1 = 0, \quad y_2 = \tfrac{1}{2}, \quad y_3 = \tfrac{1}{2}, \quad y_4 = 0.}$

Then: $y_1 + (1-y_2) + y_3 = 1 \geq 1$ ✓

$(1-y_2) + (1-y_3) + y_4 = 1 \geq 1$ ✓

But

$$y_1 + (1-y_2) + y_4 = \tfrac{1}{2} < 1.$$