Vibrations of a variable string

Consider the problem

\[
\frac{1}{(1 + x)^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 < x < 1, \quad t > 0, \quad (1)
\]

\[
u(x, 0) = f(x),
\]

\[
\frac{\partial u}{\partial t}(x, 0) = 0,
\]

\[
u(0, t) = 0,
\]

\[
u(1, t) = 0.
\]

The solution to this problem approximates the motion of a string whose linear density is proportional to \((1 + x)^{-2}\).

**The eigenvalue problem.** Separation of variables \((u = X(x)T(t))\) leads to the two equations

\[
X'' + \frac{\lambda}{(1 + x)^2} X = 0, \quad (2)
\]

and

\[
T'' + \lambda T = 0. \quad (3)
\]

For \(X(x)\) we obtain the boundary conditions

\[
X(0) = X(1) = 0.
\]

This is an eigenvalue problem as was discussed in class. To find the eigenvalues, we note that (2) has solutions of the form \((1 + x)^a\), where

\[
a(a - 1) + \lambda = 0.
\]

That is,

\[
a = \frac{1}{2} \left( 1 \mp \sqrt{1 - 4\lambda} \right).
\]

To satisfy \(X(0) = 0\), we choose

\[
X = (1 + x)^{\frac{1}{2}(1 + \sqrt{1 - 4\lambda})} - (1 + x)^{\frac{1}{2}(1 - \sqrt{1 - 4\lambda})}.
\]

The condition \(X(1) = 0\) then becomes

\[
2^{\frac{1}{2}(1 + \sqrt{1 - 4\lambda})} - 2^{\frac{1}{2}(1 - \sqrt{1 - 4\lambda})} = 0,
\]
or

\[2^{1 - 4\lambda} = 1.\]

If \(\lambda < \frac{1}{4}\) so that \(\sqrt{1 - 4\lambda}\) is real, this equation clearly has no solution. If \(\lambda = \frac{1}{4}\), the two solutions are no longer linearly independent. In fact for \(\lambda = \frac{1}{4}\), two independent solutions are \((1 + x)^{1/2}\) and \((1 + x)^{1/2}\ln(1 + x)\). The condition at \(x = 0\) is satisfied by the latter, but not it does not vanish at \(x = 1\). Thus \(\lambda = \frac{1}{4}\) is not an eigenvalue.

For \(\lambda > \frac{1}{4}\), the square root becomes imaginary. We can still obtain two solutions by the method of setting

\[(1 + x)^{1/2} e^{i\sqrt{4\lambda - 1} \ln(1 + x)} = (1 + x)^{1/2} e^{i\sqrt{4\lambda - 1} \ln(1 + x)}\]

\[= (1 + x)^{1/2} \left[ \cos \left( \sqrt{\lambda - \frac{1}{4}} \ln(1 + x) \right) + i \sin \left( \sqrt{\lambda - \frac{1}{4}} \ln(1 + x) \right) \right].\]

Two independent solutions of (2) are given by the real and imaginary parts of this expression; that is, by

\[(1 + x)^{1/2} \cos \left( \sqrt{\lambda - \frac{1}{4}} \ln(1 + x) \right) \quad \text{and} \quad (1 + x)^{1/2} \sin \left( \sqrt{\lambda - \frac{1}{4}} \ln(1 + x) \right).\]

To make \(X(0) = 0\) we put

\[X(x) = (1 + x)^{1/2} \sin \left( \sqrt{\lambda - \frac{1}{4}} \ln(1 + x) \right).\]

The condition \(X(1) = 0\) gives

\[2^{1/2} \sin \left( \sqrt{\lambda - \frac{1}{4}} \ln(2) \right) = 0.\]

Thus, \(\sqrt{\lambda - \frac{1}{4}} \ln 2\) must be an integral multiple of \(\pi:\ \sqrt{\lambda - \frac{1}{4}} \ln 2 = n\pi,\) or

\[\lambda_n = \frac{n^2 \pi^2}{(\ln 2)^2} + \frac{1}{4}, \quad n = 1, 2, 3, \ldots\]

These then, are the eigenvalues. The corresponding eigenfunctions are

\[X_n(x) = (1 + x)^{1/2} \sin \left( n\pi \frac{\ln(1 + x)}{\ln 2} \right).\]
The expansion problem. By the results on Sturm-Liouville problems (Strauss, sections 5.4, 11.3), these eigenfunctions are complete. We have the general Fourier series

\[ f(x) \sim \sum_{n=1}^{\infty} A_n X_n(x), \]

where

\[ A_n = \frac{\int_0^1 f(x)(1 + x)^{-3/2} \sin \left( n \pi \frac{\ln (1 + x)}{\ln 2} \right) dx}{\int_0^1 (1 + x)^{-1} \sin^2 \left( n \pi \frac{\ln (1 + x)}{\ln 2} \right) dx} = \frac{2}{\ln 2} \int_0^1 f(x)(1 + x)^{-3/2} \sin \left( n \pi \frac{\ln (1 + x)}{\ln 2} \right) dx. \]

Note that as an example of the Sturm-Liouville equation

\[ ((p(x)X')' + (\lambda r(x) - q(x))X) = 0, \]

the weight function \( r(x) = (1 + x)^{-2} \) in (2). The general Fourier series converges absolutely and uniformly if \( f(0) = f(1) = 0 \) and \( \int_0^1 (f')^2 dx \) is finite.

Since equation (3) with \( T(0) = 1, T'(0) = 0 \) has the solution \( \cos(\sqrt{\lambda} t) \), the problem (1) has the formal solution

\[ u(x, t) = \sum_{n=1}^{\infty} A_n \cos \left( \sqrt{\frac{n^2 \pi^2}{(\ln 2)^2} + \frac{1}{4}} t \right) (1 + x)^{1/2} \sin \left( n \pi \frac{\ln (1 + x)}{\ln 2} \right). \]

This series converges uniformly, and hence satisfies the initial and boundary conditions, when the series for \( f(x) \) converges uniformly. However, to assure continuous derivatives and satisfaction of the differential equation, we need to assume that the series for \( f'' \) converges uniformly. That is, we must suppose that \( f \) and its first two derivatives are continuous, that \( f(0) = f(1) = 0, f''(0) = f''(1) \), and that \( \int_0^1 (f'')^2 dx \) is finite.