

A

Name:

Section Time:

Multivariable Calculus and Matrix Algebra, MATH2010

First Exam, Thursday, February 7, 2002.

You may use one sheet of handwritten notes, but no other sources. The exam consists of three questions, and lasts fifty minutes. Please work all three problems. Please show all work clearly and in reasonable detail. Answers without appropriate supporting work or requested explanations may not receive full credit. No calculators are allowed.

Q1	
Q2	
Q3	
Total	
Grade	

1. (40+10 points) Let $F(x, y, z) := (2x - 3yz^2)^3$.

(a) (20 points) The equation $F(x, y, z) = -1$ defines z implicitly as a function of x and y . What are the first partial derivatives of z with respect to x and y ?

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

$$\frac{\partial F}{\partial x} = 6(2x - 3yz^2)^2$$

$$\frac{\partial F}{\partial y} = -9z^2(2x - 3yz^2)^2$$

$$\frac{\partial F}{\partial z} = -18yz(2x - 3yz^2)^2$$

S. ~~FF~~ $\frac{\partial z}{\partial x} = \frac{+1}{3yz}$ $\frac{\partial z}{\partial y} = -\frac{z}{2y}$

- (b) (20 points) Give an equation of the tangent plane to the surface $F(x, y, z) = -1$ at the point $(1, 1, 1)$.

$$\underbrace{-6}_{\frac{\partial F}{\partial x}}(x-1) + \underbrace{9}_{\frac{\partial F}{\partial y}}(y-1) + \underbrace{18}_{\frac{\partial F}{\partial z}}(z-1) = 0$$

- (c) (Extra credit: 10 points) Use the result of part (1a) to derive another equation of the tangent plane in part (1b). (Hint: We have z given implicitly as $z = f(x, y)$. We don't know $f(x, y)$ explicitly, but we do know its partial derivatives, from part (1a).)

$$\frac{1}{3}(x-1) - \frac{1}{2}(y-1) - (z-1) = 0$$

$\frac{\partial f}{\partial x}$ $\frac{\partial f}{\partial y}$

Note that this is the same plane as in (b).

2. (30 points) Let $f = (t^3 - 6t^2 + 9t)(s^2 - 2s + 3)$.

(a) (15 points) Show that $s = 1, t = 1$ is a critical point of f .

$$\frac{\partial f}{\partial s} = (t^3 - 6t^2 + 9t)(2s - 2) = 0 \quad \checkmark \text{ if } s=1, t=1$$

$$\frac{\partial f}{\partial t} = (3t^2 - 12t + 9)(s^2 - 2s + 3) = 0 \quad \checkmark \text{ if } s=1, t=1.$$

So our point is a critical point.

- (b) (15 points) Use the second order conditions to determine whether the critical point $s = 1, t = 1$ is a local minimum, a local maximum, or a saddle point.

$$\frac{\partial^2 f}{\partial s^2} = 2(t^3 - t^2 + 9t) = 8 \text{ at } s=1, t=1$$

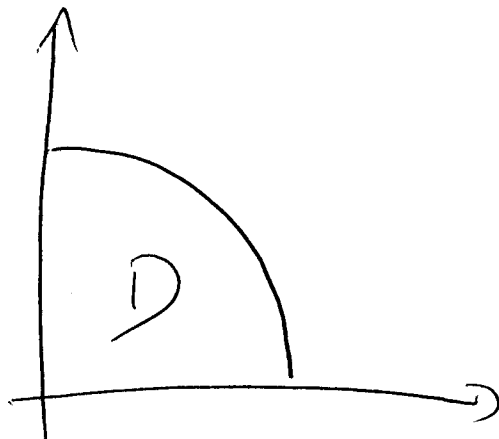
$$\frac{\partial^2 f}{\partial t^2} = (t-12)(s^2 - 2s + 3) = -12 \text{ at } s=1, t=1.$$

$$\frac{\partial^2 f}{\partial s \partial t} = (3t^2 - 12t + 9)(2s - 2) = 0 \text{ at } s=1, t=1.$$

$$\text{So } D = 8(-12) - 0 = -96 < 0, \text{ and } \frac{\partial^2 f}{\partial s^2} > 0$$

Thus, we have a saddle point.

3. (30 points) A lamina sits in the first quadrant bounded by the two axes and by the curve $x^2 + y^2 = 1$. The density of the lamina is $\rho(x, y) = x^2 + y^2 + 2$. Find the center of mass of the lamina. (Hint: The mass of the lamina is $\frac{5\pi}{8}$.)



$$\begin{aligned}
 M_y &= \iint_D x \rho(x, y) \, dA \\
 &= \int_{\theta=0}^{\pi/2} \int_{r=0}^1 r \cos \theta (r^2 + 2) \, r \, dr \, d\theta, \quad \text{using polar coordinates.} \\
 &= \int_{\theta=0}^{\pi/2} \cos \theta \left[\frac{r^5}{5} + \frac{2r^3}{3} \right]_0^1 \, d\theta \\
 &= \left(\frac{1}{5} + \frac{2}{3} \right) \left[\sin \theta \right]_0^{\pi/2} \\
 &= \frac{13}{15}
 \end{aligned}$$

$$\text{So } \bar{x} = \frac{13}{15} \bigg/ \frac{5\pi}{8} = \boxed{\frac{8 \cdot 13}{5 \cdot 15\pi}}$$

By symmetry, $\boxed{\bar{y} = \bar{x}}$