1. (25 points) Determine the values of $\sigma$ for which the boundary value problem

$$\begin{align*}
X'' + \sigma X &= 0 \\
X'(0) &= 0 = X(\pi)
\end{align*}$$

has a nonzero solution, and find the corresponding solutions.

**Case $\sigma = \lambda^2 > 0$**

$$X(x) = a \cos \lambda x + b \sin \lambda x ; \quad X'(x) = -a \lambda \sin \lambda x + b \lambda \cos \lambda x$$

$$\begin{align*}
O &= X'(0) = b \lambda \Rightarrow b = 0 \\
O &= X(\pi) = a \cos \lambda \pi \Rightarrow \pi \lambda = \left(n + \frac{1}{2}\right) \pi \\
X_n(x) &= a_n \cos \left(n + \frac{1}{2}\right) \lambda x
\end{align*}$$

**Case $\sigma = 0$**

$$X(x) = Ax + B ; \quad X'(x) = A$$

$$\begin{align*}
O &= X'(0) = A \\
O &= X(\pi) = 0
\end{align*}$$

no nonzero solutions

**Case $\sigma = -\lambda^2 < 0$**

$$X(x) = a e^{\lambda x} + be^{-\lambda x} ; \quad X'(x) = a \lambda e^{\lambda x} - b \lambda e^{-\lambda x}$$

$$\begin{align*}
O &= X'(0) = a \lambda - b \lambda \\
O &= X(\pi) = a e^{\lambda \pi} + be^{-\lambda \pi} \Rightarrow a = b e^{-2\lambda \pi} \\
O &= b e^{-2\lambda \pi} - b
\end{align*}$$

no nonzero solutions never zero

$$\begin{align*}
\sigma &= \left(\frac{n + \frac{1}{2}}{\pi}\right)^2 \\
X_n(x) &= a_n \cos \left[\left(n + \frac{1}{2}\right) \pi x\right]
\end{align*}$$

$$\sigma = \quad , \quad X(x) = \quad$$
2. (30 points) The following equations govern the steady-state temperature distribution in a rectangular plate subject to certain boundary conditions.

\[ u_{xx} + u_{yy} = 0 \]

\[ u(0, y) = u(\pi, y) \quad \text{for} \quad 0 < y < 1 \]

\[ u(x, 0) = 0 \quad \text{for} \quad 0 < x < \pi \]

\[ u(x, 1) = 5 \sin(2x) \quad \text{for} \quad 0 < x < \pi. \]

In the process of solving the periodic initial-boundary value problem, you look for solutions of the form \( u(x, y) = X(x)Y(y) \), and you find that \( X \) satisfies the equation \( X'' + \sigma X = 0 \) with zero boundary conditions. This equation has nonzero solutions only when \( \sigma = n^2, n = 1, 2, \ldots \), and for these values of \( \sigma \), \( X \) is of the form \( X_n(x) = b_n \sin nx \). Use this information to solve the rest of the problem, i.e., find a solution \( u \) that satisfies the full boundary value problem given above.

\[
X'' + \sigma X = 0 \quad \Rightarrow \quad X'' + \frac{\sigma}{X} = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\sigma
\]

\[
Y'' - \sigma Y = 0 \quad , \quad \sigma = n^2
\]

\[
0 = X(0)Y(0) \quad \Rightarrow \quad Y'(0) = 0
\]

\[
Y(y) = c_1 e^{\sigma y} + c_2 e^{-\sigma y}
\]

\[
Y'(y) = c_1 \sigma e^{\sigma y} - c_2 \sigma e^{-\sigma y}
\]

\[
0 = Y(0) = c_1 - c_2 \quad \Rightarrow \quad c_1 = c_2
\]

\[
Y_n(y) = c_n \cosh ny \quad \left( = c_n \frac{e^{ny} + e^{-ny}}{2} \right)
\]

\[
u_n(x, y) = \sinh nx
\]

\[
u(x, y) = \sum_{n=1}^{\infty} c_n \cosh n y \sin nx
\]

\[
5 \sin 2x = u(x, 1) = \sum_{n=1}^{\infty} c_n \cosh n \sin nx \quad \Rightarrow \quad c_2 \cosh 2 = 5 \quad \text{, all other } c_i \text{'s are 0}
\]

\[
c_2 = \frac{5}{\cosh 2}
\]

\[
u(x, y) = \frac{5}{\cosh 2} \cosh 2y \sin 2x
\]
3. (25 points) Suppose a spring-mass system is governed by the initial-value problem

\[ y'' + 5y = 3 \cos \omega t \]

\[ y(0) = 0; \quad y'(0) = 1 \]

Fill in the following blanks with the best answers.

a) In using the method of undetermined coefficients to find a particular solution of the differential equation, the best trial solution is \( A \cos \omega t \).

b) The natural (angular) frequency of the system is \( \sqrt{5} \) radians/sec.

c) When \( \omega = 10 \), the graph that best describes the behavior of the system is 5 or 6.

d) Resonance occurs when \( \omega = \sqrt{5} \) radians/sec.

e) At resonance, the graph (below) that best describes the behavior of the system is 6.
4. (20 points) Sketch the function \( f(x) \) to which the Fourier series converges. Pay special attention to points of discontinuity.

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2nx
\]

where

\[
a_n = \frac{2}{\pi} \int_{0}^{\frac{\pi}{4}} x \cos 2nx dx
\]

\( f \) should be even and periodic. Compare with formulas

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}
\]

Here \( 2nx = \frac{n\pi x}{2} \Rightarrow L = \frac{\pi}{2} \), so

\[
a_n = \frac{2}{\pi} \int_{0}^{\pi/2} f(x) \cos \frac{n\pi x}{L} dx
\]

Thus, \( f(x) = \left\{ \begin{array}{ll} x & \text{for } 0 < x < \frac{\pi}{4} \\ 0 & \text{in rest of interval} \end{array} \right. \)

\[
f\left( \frac{\pi}{4} \right) = \frac{\pi}{4} \int_{x=\pi/4}^{\pi/2} \frac{\pi^2}{16}
\]

Points for grading:

a) \( L = \frac{\pi}{2} \)

b) \( f \) even

c) \( f \) periodic

d) \( f = \frac{\pi}{4} x \) on \( 0 < x < \frac{\pi}{4} \)

e) \( f = 0 \) on \( \frac{\pi}{4} < x < \frac{\pi}{2} \)

f) behavior at discontinuities