Sequences

A sequence is an ordered list of terms. To determine if a sequence converges or diverges, one examines the behavior of \( \lim_{x \to \infty} a_n \).

Write out the first 5 terms of each sequence

\[
\left\{ \frac{2n}{n + 4} \right\}
\]

**Solution.** Be sure to start your sequence at \( n = 1 \).

\[
\frac{2}{5}, \frac{2}{3}, \frac{6}{7}, 1, \frac{10}{9}, \ldots
\]

\[a_n = \cos \left( \frac{n\pi}{6} \right)\]

**Solution.** Start your sequence at \( n = 1 \) and evaluate your trigonometric functions.

\[
\cos \left( \frac{\pi}{6} \right), \cos \left( \frac{\pi}{3} \right), \cos \left( \frac{\pi}{2} \right), \cos \left( \frac{2\pi}{3} \right), \cos \left( \frac{5\pi}{6} \right), \ldots
\]

Thus \(\sqrt{3} \frac{3}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\sqrt{3} \frac{3}{2}, \ldots\)

In the following sequence, determine a formula for the general term \( a_n \) assuming that the pattern of the first few terms continues.

\[
\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \ldots
\]

**Solution.** Make sure to find a \( n^{th} \) term definition and not a recursive formula.

\[
a_n = \frac{(-1)^n}{(n + 1)^2}
\]
Determine whether or not the following sequences converge or diverge. Find the limit of each sequence that converges.

**Solution.** To determine convergence of each sequence, look at \( \lim_{n \to \infty} a_n \).

\[
a_n = \frac{3 + 5n^2}{n + n^2}
\]

**Solution.** When determining \( \lim_{n \to \infty} a_n \), look at the ratio of the highest powers in the numerator and denominator.

\[
\lim_{n \to \infty} \frac{3 + 5n^2}{n + n^2} = \lim_{n \to \infty} \frac{5n^2}{n^2} = 5
\]

Thus the sequence converges to 5.

\[
a_n = \frac{3^{n+2}}{5^n}
\]

**Solution.** When determining \( \lim_{n \to \infty} a_n \), simplify \( a_n \) before taking the limit.

\[
\lim_{n \to \infty} \frac{3^{n+2}}{5^n} = \lim_{n \to \infty} \frac{3^2 \cdot 3^n}{5^n}
\]

\[
= 3^2 \lim_{n \to \infty} \left( \frac{3}{5} \right)^n
\]

\[
= 3^2 \cdot 0 = 0
\]

Thus the sequence converges to 0.

\[
a_n = \cos(n\pi)
\]

**Solution.** When determining \( \lim_{n \to \infty} a_n \), one notes that the sequence oscillates between \(-1\) and 1. Specifically, the sequences is

\[-1, 1, -1, 1, -1, 1, -1, 1, \ldots\]

Thus the sequence diverges.
A series is an infinite sum of terms. In this section we have 3 methods to determine convergence or divergence of series: 1) Sequence of Partial Sums and telescoping series, 2) Geometric Series, and 3) Divergence Test or $n^{th}$ term test.

Find the sum of the convergent series

$$
\sum_{n=1}^{\infty} \frac{1}{n(n+1)}
$$

**Solution.** Using Partial Fraction Decomposition one writes $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$ and solves to find that $A = 1$ and $B = -1$. Thus

$$
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)
$$

This is a telescoping sequence and if the sequence converges, we can find the sum using Sequence of Partial Sums.

$$
S_1 = 1 - \frac{1}{2}
$$

$$
S_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3}
$$

$$
S_3 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4}
$$

$$
\cdots
$$

$$
S_n = 1 - \frac{1}{n+1}
$$

Taking the limit of the $n^{th}$ term definition for $S_n$

$$
\lim_{n \to \infty} \left( 1 - \frac{1}{n+1} \right) = 1
$$

Thus $\{S_n\} \to 1$ and

$$
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \text{ converges and has sum } = 1.
$$
Express the decimal $0.07575$ as a geometric series and write its sum as the ratio of two integers.

**Solution.** Expanding the decimal and writing it as a geometric series we get

$$0.07575 = 0.075 + 0.00075 + 0.0000075 + \cdots$$

$$= \frac{75}{1000} + \frac{75}{1000} \left( \frac{1}{100} \right) + \frac{75}{1000} \left( \frac{1}{100} \right)^2 + \cdots + \frac{75}{1000} \left( \frac{1}{100} \right)^n + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{75}{1000} \left( \frac{1}{100} \right)^n = \sum_{n=0}^{\infty} \frac{3}{40} \left( \frac{1}{100} \right)^n$$

Finding the sum gives us

$$\text{sum} = \frac{1}{1 - r} = \frac{\frac{3}{40}}{1 - \frac{1}{100}} = \frac{3}{40} \cdot \frac{100}{99} = \frac{5}{66}$$

Thus

$$0.07575 = \frac{5}{66}.$$ 

Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{3n - 1}{2n + 1}$$

**Solution.** Since this series is not telescoping or geometric, we look use the $n^{th}$ term test or the Divergence test.

$$\lim_{n \to \infty} \frac{3n - 1}{2n + 1} = \frac{3}{2} \neq 0$$

Therefore by the Divergence Test the series diverges.
An important result regarding p-series, \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) is convergent if \( p > 1 \) and divergent if \( p \leq 1 \).

Alternating series, \( \sum_{n=1}^{\infty} (-1)^{n-1}b_n \) where \( b_n > 0 \) must satisfy the following two conditions in order to converge:

- \( b_{n+1} \leq b_n \) for all \( n \)
- \( \lim_{n \to \infty} b_n = 0 \)

Determine the convergence or divergence of the following series:

\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}} \]

**Solution.** This series behaves like \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) or \( \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \). Using the p-test, we have that \( p = \frac{1}{2} < 1 \) therefore our series **DIVERGES**.

\[ \sum_{n=1}^{\infty} \frac{1}{n^2 - 4n + 5} \]

**Solution.** This series behaves like \( \sum_{n=1}^{\infty} \frac{1}{n^2} \). Using the p-test, we have that \( p = 2 > 1 \) therefore our series **CONVERGES**.
\[ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{1/n}}{n} \]

**Solution.** Note that this is an alternating series where \( b_n = \frac{e^{1/n}}{n} \). The easiest way to determine if \( b_{n+1} \leq b_n \) is to look at the function \( f(x) = \frac{e^{1/x}}{x} \) and determine if \( f'(x) \leq 0 \).

\[
f'(x) = - \frac{1}{x^2} e^{1/x} + \frac{1}{x} e^{1/x} \left(- \frac{1}{x^2}\right)
= - \left( \frac{1}{x^2} + \frac{1}{x^3} \right) e^{1/x}
< 0 \text{ for all } x \in \mathbb{R}
\]

Thus we have that \( b_{n+1} \leq b_n \). Next we check if \( \lim_{n \to \infty} b_n = 0 \).

\[
\lim_{n \to \infty} \frac{e^{1/n}}{n} = \frac{e^0}{\infty} = \frac{1}{\infty} = 0
\]

Therefore by the alternating series test we have shown that the series **CONVERGES**. 

*Note that the Ratio test from 11.6 fails to give a result for this problem.*

\[ \sum_{n=1}^{\infty} (-\frac{n}{3})^n \]

**Solution.** Note that this is an alternating series where \( b_n = \left(\frac{n}{3}\right)^n \). If we first look at \( \lim_{n \to \infty} b_n \) we see that

\[
\lim_{n \to \infty} \left(\frac{n}{3}\right)^n = \infty
\]

since we have that \( \lim_{n \to \infty} \frac{n}{3} = \infty \). Thus by the alternating series test we have shown that the series **DIVERGES**. *Note that using the Root test from 11.6 gives you the same result.*
Ratio Test look at:  
\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \]

Root Test look at:  
\[ \lim_{n \to \infty} \sqrt[n]{|a_n|} = L \]

For either test, if \( L < 1 \) then converges  
if \( L > 1 \) then diverges  
if \( L = 1 \) then undetermined ... could converge or could diverge.

Write the equivalent series with the index of summation beginning at \( n = 0 \).

\[ \sum_{n=2}^{\infty} \frac{2^n}{(n-2)!} \]

**Solution.** Recalling that \( 0! = 1 \) we see that the series can be expanded as

\[ \sum_{n=2}^{\infty} \frac{2^n}{(n-2)!} = 4 + \frac{8}{1} + \frac{16}{1 \cdot 2} + \frac{32}{1 \cdot 2 \cdot 3} + \cdots \]

We can rewrite the summation to start at \( n = 0 \) by replacing \( n \) with \( n + 2 \) in \( a_n \) as follows

\[ \sum_{n=0}^{\infty} \frac{2^{n+2}}{((n+2)-2)!} = \sum_{n=0}^{\infty} \frac{2^{n+2}}{n!} \]

Determine the convergence or divergence of the following series:

\[ \sum_{n=1}^{\infty} \frac{3^n}{n!} \]

**Solution.** For this series we use the Ratio test to determine our answer

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} \right| \]

\[ = \lim_{n \to \infty} \left| \frac{3}{n+1} \right| \]

\[ = 0 < 1 \]

Thus by the Ratio test we find that our series **CONVERGES**
Determine the convergence or divergence of the following series:

\[ \sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n} \]

**Solution.** We note that for this series both the numerator and denominator are raised to the power \( n \), thus we use the Root test to determine our answer

\[
\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \sqrt[n]{\left|\frac{-1}{\ln (n)}\right|^n}
\]

\[= \lim_{n \to \infty} \left|\frac{-1}{\ln (n)}\right|\]

\[= \lim_{n \to \infty} \frac{1}{\ln (n)}\]

\[= 0 < 1\]

Thus by the Root test we find that our series **CONVERGES**.
A Power series is a series that includes powers of $x$ or $(x - c)$. Power series are written as $\sum a_n x^n$ or $\sum a_n (x - c)^n$.

Find the Interval and Radius of convergence for the power series given below. Be sure to test the convergence at the endpoints of the interval:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x - 5)^n}{n 5^n}$$

**Solution.** We use the Ratio Test for fixed $x$ where $x \neq 5$ and find that

$$L = \lim_{n \to \infty} \left| \frac{(-1)^{n+2}(x - 5)^{n+1}}{(n + 1) 5^{n+1}} \cdot \frac{n 5^n}{(-1)^{n+1}(x - 5)^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{n}{5(n + 1)} \cdot (x - 5) \right| = \left( \lim_{n \to \infty} \left| \frac{n}{5n + 5} \right| \right) \left( \lim_{n \to \infty} |x - 5| \right) = \frac{1}{5} |x - 5|$$

For convergence we need that $L < 1$ which gives us

$$\frac{|x - 5|}{5} < 1$$

$$|x - 5| < 5$$

$$-5 < x - 5 < 5$$

$$0 < x < 10$$

The value for $R$, the radius of convergence, is found by determining the length of the interval and dividing by 2, which in this case gives us that $R = 5$. We now need to determine the convergence at the endpoints of the interval found above to complete the solution for the interval of convergence.

Determining convergence when $x = 0$ we have that the power series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-5)^n}{n 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-5)^n}{n 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} = \sum_{n=1}^{\infty} \frac{-1}{n}$$

Applying the $p$-test we see that $p = 1$, thus at $x = 0$, the power series diverges.

Determining convergence when $x = 10$ we have that the power series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(5)^n}{n 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(5)^n}{n} = \sum_{n=1}^{\infty} (-1)^n + \frac{1}{n}$$

Applying the alternating series test we verify that $b_{n+1} < b_n$, or $\frac{1}{n + 1} < \frac{1}{n}$ and that

$$\lim_{n \to \infty} \frac{1}{n} = 0,$$

thus we have that at $x = 10$ the power series converges.

Thus we have shown that

$$0 < x \leq 10; \quad R = 5.$$
\[ \sum_{n=1}^{\infty} \frac{(2n)!x^n}{n!} \]

**Solution.** We use the Ratio Test for fixed \( x \) where \( x \neq 0 \) and find that

\[
L = \lim_{n \to \infty} \left| \frac{(2(n+1))!(x^{n+1})}{(n+1)!} \cdot \frac{n!}{(2n)!(x)^n} \right| \\
= \lim_{n \to \infty} \left| \frac{(2n + 2)(2n + 1)}{(n + 1)} \cdot x \right| = \left( \lim_{n \to \infty} \left| \frac{4n^2 + 6n + 2}{n + 1} \right| \right) \cdot \left( \lim_{n \to \infty} |x| \right) \to \infty
\]

For convergence we need that \( L < 1 \) and since our limit is not finite, the only way we can force \( L < 1 \) is to let \( x = 0 \). By letting \( x = 0 \) our series becomes \( \sum_{n=1}^{\infty} 0 \) and we have that

\[
x = 0 ; \quad R = 0.
\]
MATH 1020 WORKSHEET 11.9
Geometric Power Series

Functions of the form \( f(x) = \frac{a}{1-r} \) where \( a \) is any polynomial in \( x \) or a constant and \( r \) is a polynomial in \( x \) can be rewritten as a power series \( f(x) = \sum_{n=0}^{\infty} ar^n \) as long as \( |r| < 1 \)

Determine the geometric power series for the following functions centered at the indicated \( c \) value.

\[ f(x) = \frac{2}{1-x^2} \quad c = 0 \]

**Solution.** The function \( f(x) \) is already in the correct form where we have \( 1- \) in the denominator. Thus we know that \( a = 2 \) and \( r = x^2 \). Thus we have that if \( |x^2| < 1 \) or if \(-1 < x < 1\) then

\[
\frac{2}{1-x^2} = \sum_{n=0}^{\infty} 2(x^2)^n = \sum_{n=0}^{\infty} 2x^{2n}
\]

\[ f(x) = \frac{3}{2x-1} \quad c = 2 \]

**Solution.** To determine a geometric power series for \( f(x) \) centered at \( c = 2 \) we must first rewrite the denominator in terms of \( (x-2) \). Once this is done we can then manipulate the function so that we have a \( 1- \) in the denominator. The calculations to do this follow

\[
\frac{3}{2x-1} = \frac{3}{2(x-2) + 1} = \frac{3}{2(x-2) + 3}
\]

now we have \( (x-2) \) in the denominator

\[
\frac{3}{3 + 2(x-2)} = \frac{1}{1 + \frac{2}{3}(x-2)} = \frac{1}{1 - \left(-\frac{2}{3}(x-2)\right)}
\]

We have found that \( a = 1 \) and \( r = -\frac{2}{3}(x-2) \). Determining the interval of convergence we find that

\[
\left| -\frac{2}{3}(x-2) \right| < 1
\]

\[-1 < \frac{2}{3}(x-2) < 1 \]

\[-\frac{3}{2} < x-2 < \frac{3}{2} \]

\[ \frac{1}{2} < x < \frac{7}{2} \]

Thus we have that if \( \frac{1}{2} < x < \frac{7}{2} \) then
\[ \frac{3}{2x-1} = \sum_{0}^{\infty} \left( \frac{-2}{3} (x-2)^n \right) = \sum_{0}^{\infty} (-1)^n \left( \frac{2}{3} \right)^n (x-2)^n \].

\[ f(x) = \ln (1 - x^2) = \int \frac{1}{1+x} \, dx - \int \frac{1}{1-x} \, dx \]

**Solution.** To solve this problem, we must replace \( \frac{1}{1+x} \) and \( \frac{1}{1-x} \) with geometric power series before we integrate. We find that

\[ \frac{1}{1+x} = \frac{1}{1-(-x)} \quad \text{thus} \quad a = 1 \quad \text{and} \quad \frac{1}{1-x} \quad \text{thus} \quad a = 1 \]

\[ r = -x \quad \text{if} \quad | -x | < 1 \]

\[ |x| < 1 \]

then \( \frac{1}{1+x} = \sum_{0}^{\infty} (-x)^n \)

then \( \frac{1}{1-x} = \sum_{0}^{\infty} (x)^n \)

From this we have that if \(-1 < x < 1\) we can write

\[ \ln (1 - x^2) = \int \sum_{0}^{\infty} (-1)^n x^n \, dx - \int \sum_{0}^{\infty} x^n \, dx \]

\[ = C + \sum_{0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} - \sum_{0}^{\infty} \frac{x^{n+1}}{n+1} \]

\[ = C + \sum_{0}^{\infty} \frac{[(-1)^n - 1]}{n+1} x^{n+1} \]

To solve for \( C \) we can evaluate both sides of the equation at \( x = 0 \) (we pick \( x = 0 \) since this makes every term in the power series equal to zero). Thus we have that \( \ln (1) = C + 0 \) or that \( C = 0 \). Thus

\[ \ln (1 - x^2) = \sum_{0}^{\infty} \frac{[(-1)^n - 1]}{n+1} x^{n+1} \]

Note that if we expand the power series we find that all the odd terms are equal to zero and only the even terms remain.

\[ \ln (1 - x^2) = \sum_{0}^{\infty} \frac{[(-1)^n - 1]}{n+1} x^{n+1} \]

\[ = 0 + \frac{-2}{2} x^2 + \frac{0}{3} x^3 + \frac{-2}{4} x^4 + \frac{0}{5} x^5 + \frac{-2}{6} x^6 \cdots \]

\[ = -x^2 - \frac{1}{2} x^4 - \frac{1}{3} x^6 \cdots \]

\[ = \sum_{0}^{\infty} \frac{-x^{2n}}{n} \]
A Taylor series for a function \( f(x) \) centered at \( c \) has the form

\[
f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n.
\]

If \( c = 0 \), this is also called a Maclaurin Series for \( f(x) \).

Find the Taylor series for \( f(x) = e^{-2x} \) centered at \( c = 0 \).

**Solution.** To determine the Taylor series we first create a table of information with derivatives of \( f(x) \) and the derivatives evaluated at \( c = 0 \). Then we can put all the information into the form of the Taylor series.

\[
\begin{array}{|c|c|c|}
\hline
n & f^{(n)}(x) & f^{(n)}(0) \\
\hline
0 & e^{-2x} & e^0 = 1 \\
1 & (-2)e^{-2x} & -2e^0 = -2 \\
2 & -2(-2e^{-2x}) & 4e^0 = 4 \\
3 & (-2)^3e^{-2x} & -8e^0 = -8 \\
\vdots & \vdots & \vdots \\
n & \vdots & (-1)^n2^n \\
\hline
\end{array}
\]

Thus we have that

\[
e^{-2x} = 1 - 2x + \frac{4}{2!}x^2 - \frac{8}{3!}x^3 \cdots + \frac{(-1)^n2^n}{n!}x^n + \cdots
\]

Thus our answer is

\[
e^{-2x} = \sum_{n=0}^{\infty} \frac{(-1)^n2^n}{n!}x^n.
\]

Find the Taylor series for \( f(x) = \sin x \) centered at \( c = \pi/4 \).

**Solution.** To determine the Taylor series we first create a table of information with derivatives of \( f(x) = \sin x \) and the derivatives evaluated at \( c = \pi/4 \). Then we can put all the information into the form of the Taylor series.

\[
\begin{array}{|c|c|c|}
\hline
n & f^{(n)}(x) & f^{(n)}(\pi/4) \\
\hline
0 & \sin x & \sin (\pi/4) = \frac{1}{\sqrt{2}} \\
1 & \cos x & \cos (\pi/4) = \frac{1}{\sqrt{2}} \\
2 & -\sin x & -\frac{1}{\sqrt{2}} \\
3 & -\cos x & -\frac{1}{\sqrt{2}} \\
4 & \sin x & \frac{1}{\sqrt{2}} \\
5 & \cos x & \frac{1}{\sqrt{2}} \\
6 & -\sin x & -\frac{1}{\sqrt{2}} \\
\vdots & \vdots & \vdots \\
2n & \vdots & \vdots \\
2n + 1 & \vdots & \vdots \\
\hline
\end{array}
\]

Thus we have that

\[
\sin x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \left( x - \frac{\pi}{4} \right) - \frac{1}{\sqrt{2}2!} \left( x - \frac{\pi}{4} \right)^2 - \frac{1}{\sqrt{2}3!} \left( x - \frac{\pi}{4} \right)^3 + \frac{1}{\sqrt{2}4!} \left( x - \frac{\pi}{4} \right)^4 + \frac{1}{\sqrt{2}5!} \left( x - \frac{\pi}{4} \right)^5 + \cdots
\]

Thus we have that

\[
\sin x = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2}(2n)!} \left( x - \frac{\pi}{4} \right)^{2n} + \sum_{n=0}^{\infty} \frac{1}{\sqrt{2}(2n + 1)!} \left( x - \frac{\pi}{4} \right)^{2n+1}.
\]
Find the Taylor series centered at \( c = 0 \) (Maclaurin) for the function \( f(x) = \cosh x = \frac{1}{2} (e^x + e^{-x}) \) given the power series for \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad R = \infty \).

**Solution.** To find a power series for \( \cosh x \) we use the given power series for \( e^x \) to replace the terms \( e^x \) and \( e^{-x} \) in the definition of \( \cosh x \).

\[
\cosh x = \frac{1}{2} (e^x + e^{-x}) \\
= \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \right) \\
= \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \right) \\
= \frac{1}{2} \left( \sum_{n=0}^{\infty} \left(1 + (-1)^n\right) \frac{x^n}{n!} \right) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{2}{(2n)!} x^{2n}
\]

The last step is determined by realizing that all the odd terms are \( 1 - 1 = 0 \) and all the even terms we have \( 1 + 1 = 2 \). Thus we have that

\[
\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}
\]

Find the Taylor series centered at \( c = 0 \) (Maclaurin) for the function \( f(x) = x^2 \arctan (x^3) \) given the power series for \( \arctan (x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1} \quad R = 1 \)

**Solution.** Here we use the given series for \( \arctan (x) \) and then manipulate the summation until we get all the powers of \( x \) together.

\[
x^2 \arctan (x^3) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{2n + 1} \\
= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} x^2 x^{3(2n+1)} \\
= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} x^{2n+3} \\
= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} x^{6n+5}
\]
A Taylor polynomial approximation of degree $n$ for a function $f(x)$ centered at $c$ has the form $P_n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$. If $c = 0$, this is also called a Maclaurin polynomial approximation for $f(x)$.

The Taylor Remainder and Inequality on an interval follows

\[
\text{Given } f(x) = T_n(x) + R_n(x), \text{ then } |R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}
\]
where $M$ is the maximum value of $|f^{(n+1)}(x)|$ on the interval and $|x-a| \leq d$.

Approximate $f(x) = \sin(x)$ by a Taylor polynomial of degree $n = 4$ at $a = \frac{\pi}{6}$. Use Taylor’s Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when $0 \leq x \leq \frac{\pi}{3}$.

**Solution.** For the Taylor polynomial of degree $n = 4$ we create a chart that includes $f(x) = \sin(x)$ and its first 4 derivatives evaluated at $a = \frac{\pi}{6}$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f^{(n)}(x)$</th>
<th>$f^{(n)}(\pi/6)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\sin x$</td>
<td>$\sin(\pi/6) = \frac{1}{2}$</td>
</tr>
<tr>
<td>1</td>
<td>$\cos x$</td>
<td>$\cos(\pi/6) = \frac{\sqrt{3}}{2}$</td>
</tr>
<tr>
<td>2</td>
<td>$-\sin x$</td>
<td>$-\frac{1}{2}$</td>
</tr>
<tr>
<td>3</td>
<td>$-\cos x$</td>
<td>$-\frac{\sqrt{3}}{2}$</td>
</tr>
<tr>
<td>4</td>
<td>$\sin x$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>5</td>
<td>$\cos x$</td>
<td></td>
</tr>
</tbody>
</table>

Thus we have that

\[
T_4 = \frac{1}{2} + \frac{\sqrt{3}}{2} \left( x - \frac{\pi}{6} \right) - \frac{1}{2 \cdot 2!} \left( x - \frac{\pi}{6} \right)^2 \]
\[
- \frac{\sqrt{3}}{2 \cdot 3!} \left( x - \frac{\pi}{6} \right)^3 + \frac{1}{2 \cdot 4!} \left( x - \frac{\pi}{6} \right)^4
\]

To estimate the accuracy of the approximation $f(x) \approx T_n(x)$ on $0 \leq x \leq \frac{\pi}{3}$ we need to determine the maximum possible value of the 5th derivative of $\sin x$ on the interval. We see that $f^{(5)} = \cos x$. The maximum of $|\cos x|$, $M$, on $[0, \pi/3]$ occurs at $x = 0$ and is $M = 1$. Also, the maximum of $|x - \pi/6|$ on $[0, \pi/6]$ is $\pi/6$. Thus we have that

\[
|R_4(x)| \leq \frac{1}{5!} \left( \frac{\pi}{6} \right)^5
\]
\[
|R_4(x)| \leq \frac{\pi^5}{6^5 \cdot 5!}
\]
Approximate $f(x) = x \ln(x)$ by a Taylor polynomial of degree $n = 3$ at $a = 1$. Use Taylor’s Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when $0.5 \leq x \leq 1.5$.

**Solution.** For the Taylor polynomial of degree $n = 3$ we create a chart that includes $f(x) = x \ln(x)$ and its first 3 derivatives evaluated at $a = 1$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f^{(n)}(x)$</th>
<th>$f^{(n)}(\pi/6)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$x \ln x$</td>
<td>$1 \ln 1 = 0$</td>
</tr>
<tr>
<td>1</td>
<td>$\ln x + x \left(\frac{1}{x}\right)$</td>
<td>$0 + 1 = 1$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{x}$</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$-\frac{1}{x^2}$</td>
<td>$-\frac{1}{2}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{2}{x^3}$</td>
<td></td>
</tr>
</tbody>
</table>

Thus we have that

$$T_3 = 0 + (x - 1) + \frac{1}{2!}(x - 1)^2 - \frac{1}{3!}(x - 1)^3$$

To estimate the accuracy of the approximation $f(x) \approx T_n(x)$ on $0.5 \leq x \leq 1.5$ we need to determine the maximum possible value of the 4th derivative of $x \ln x$ on the interval. We see that $f^{(4)} = \frac{2}{x^3}$. The maximum of $|\frac{2}{x^3}|$, $M$, on $[0.5, 1.5]$ occurs at $x = 0.5$ and is $M = \frac{2}{(0.5)^3} = 16$. Also, the maximum of $|x - 1|$ on $[0.5, 1.5]$ is $1/2$. Thus we have that

$$|R_3(x)| \leq \frac{16}{4!} \left(\frac{1}{2}\right)^4$$

$$= \frac{16}{24} \cdot \frac{1}{16}$$

$$= \frac{1}{24}$$

Thus the upper bound on the error for using $T_3$ to approximate $f(x) = x \ln x$ on the interval $[0.5, 1.5]$ is

$$|R_3(x)| \leq \frac{1}{24}.$$