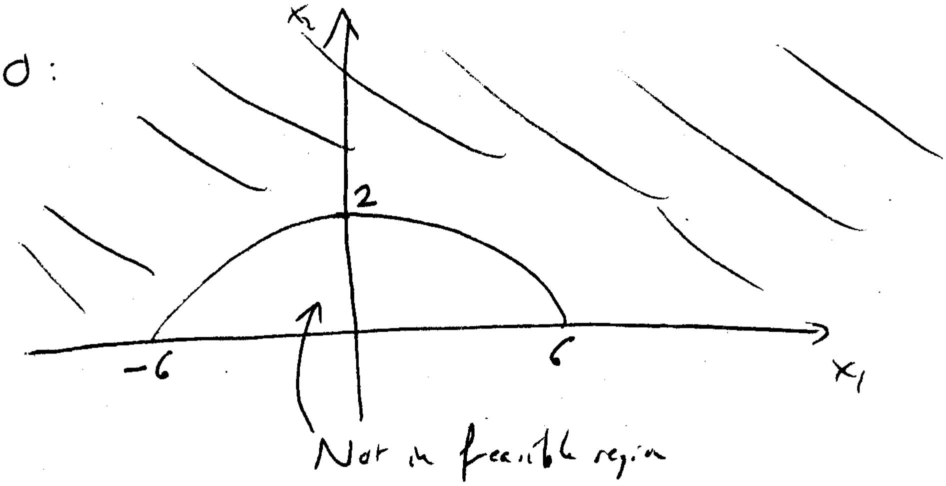
1. (a) No:



(b) $g_1(x) = -x_1^2 - (x_2+8)^2 + 100$ $\qquad \nabla g_1 = \begin{bmatrix} -2x_1 \\ -2(x_2+8) \end{bmatrix}$

$g_2(x) = -x_2$ $\qquad \nabla g_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The only point where ∇g_1 is zero is $(0,-8)$, which is infeasible.

Thus, for the gradients to be linearly dependent, both constraints must be active. But then $x_1 \neq 0$, so ∇g_1 and ∇g_2 are linearly independent.

(c) Gradient condition:

$$\begin{bmatrix} 2(x_1+3) \\ 2(x_2+12) \end{bmatrix} + u_1 \begin{bmatrix} -2x_1 \\ -2(x_2+8) \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Complementarity:
$$u_1\left(100 - x_1^2 - (x_2+8)^2\right) = 0, \quad u_2 x_2 = 0$$

Nonnegativity: $u_1, u_2 \geq 0$.

Feasibility: $x_1^2 + (x_2+8)^2 \geq 100, \quad x_2 \geq 0.$

(d) $x = (6,0)$: Both constraints are active,
Gradient condition is:

$$\begin{bmatrix}18\\24\end{bmatrix} + u_1\begin{bmatrix}-12\\-16\end{bmatrix} + u_2\begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Soln: $u_1 = 1.5$, $u_2 = 0$. So YES, KKT point.

$x = (-6,0)$: Both constraints are active.
Gradient condition is:

$$\begin{bmatrix}-6\\24\end{bmatrix} + u_1\begin{bmatrix}12\\-16\end{bmatrix} + u_2\begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

Soln: $u_1 = \frac{1}{2}$, $u_2 = 16$. So YES, KKT point.

(e) If $x_1^2 + (x_2+8)^2 > 100$ then must have $u_1 = 0$.
Gradient condition becomes

$$\begin{bmatrix}2(x_1+3)\\2(x_2+12)\end{bmatrix} + u_2\begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$\Rightarrow x_1 = -3$, $2(x_2+12) - u_2 = 0$

$\Rightarrow x_2 = 0,\ u_2 = 24$ or $x_2 = -12,\ u_2 = 0$ since $u_2 x_2 = 0$

But neither $(-3,0)$ nor $(-3,-12)$ is feasible.

(f) From part (e), need only look at points with $x_1^2 + (x_2+8)^2 = 100$.

From part (d), we've already considered both points when both constraints are active.

Thus, only need to look at points with $x_2 > 0$ and $x_1^2 + (x_2+8)^2 = 100$

Gradient condition becomes, since $u_2 = 0$:

$$\begin{bmatrix} 2(x_1+3) \\ 2(x_2+12) \end{bmatrix} + u_1 \begin{bmatrix} -2x_1 \\ -2(x_2+8) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad \begin{matrix} ① \\ ② \end{matrix}$$

$\frac{1}{2}(x_2+8)$ ① $- \frac{1}{2}x_1$ ② $\Rightarrow$

$$(x_1+3)(x_2+8) - x_1(x_2+12) = 0$$

$$\Rightarrow 3x_2 - 4x_1 + 24 = 0 \Rightarrow x_1 = 6 + \tfrac{3}{4}x_2 \geq 6 \text{ if } x_2 \geq 0.$$

But the only point satisfying $x_1^2 + (x_2+8)^2 = 100, x_2 \geq 0, x_1 \geq 6$ is $(6,0)$ which we already considered.

(g) Let $S = \{x \in \mathbb{R}^2 : x_1^2 + (x_2+8)^2 \geq 100, x_2 \geq 0, (x_1+3)^2 + (x_2+12)^2 \leq 1000\}$
S is compact and so the objective function $(x_1+3)^2 + (x_2+12)^2$ must attain its minimum on S, by Weierstrass's Thm.
Any point outside S has worse value than any point in S. Thus, there exists a global minimizer to our problem.
From part (b), Constraint Qualification holds at this global min, so it must be a KKT point. Thus, the global min is one of the two KKT points, $(6,0)$ or $(-6,0)$. $(6,0)$ has value $81+144=225$. $(-6,0)$ has value $9+144=153$. Thus, the global min is $\boxed{(-6,0)}$

2. (a) This is an unconstrained problem, so we need $\nabla f = 0$.

$$\nabla f = \sum u_i (x - y_i)$$

Thus, we need $\sum u_i x = \sum u_i y_i$,

that is, $x = Yu$ since $\sum u_i = 1$.

(b) Equivalent to:

$$\min_{x,z} \quad z$$
$$\text{s.t.} \quad z \geq \|x - y_i\|_2^2 \quad \text{for } i = 1, \ldots, m.$$

or:

$$\min \quad z$$
$$\text{s.t.} \quad z \geq x^T x - x^T y_i + y_i^T y_i \quad \forall i$$

or:

$$\min \quad w + x^T x$$
$$\text{s.t.} \quad w \geq -x^T y_i + y_i^T y_i \quad \forall i$$

(Here, $w = z - x^T x$)

3. (a) $L(x,\nu) = \frac{1}{2} x^T M x + \nu\left(-\frac{1}{2} x^T x + \frac{1}{2}\right)$

$$= \frac{1}{2} x^T (M - \nu I) x + \frac{1}{2}\nu$$

$$g(\nu) = \min_x L(x,\nu) = \begin{cases} -\infty & \text{if } M - \nu I \text{ has negative eval} \\ +\frac{1}{2}\nu & \text{otherwise} \end{cases}$$

$$= \begin{cases} +\frac{1}{2}\nu & \text{if } \nu \le \lambda_1 \\ -\infty & \text{otherwise.} \end{cases}$$

(b) Solution is $\nu = \lambda_1$, with value $\frac{1}{2}\lambda_1$

(c) Infimum is achieved by any x in the nullspace of $M - \lambda_1 I$, that is, by any eigenvector of M with eigenvalue λ_1.

One of the ~~solutions~~ ~~is~~ eigenvectors satisfies $\frac{1}{2} x^T x = \frac{1}{2}$, and this solves the primal problem, with value

$$\frac{1}{2} x^T M x = \frac{1}{2} x^T (\lambda_1 x) = \frac{1}{2}\lambda_1 x^T x = \frac{1}{2}\lambda_1,$$

ie, the same as the dual value.

4. (a) If $p^T w = \alpha$ separates the sets, then $a_i^T p > \alpha > b_j^T p$ for any a_i, b_j.

Equivalently,
$$\left.\begin{array}{l} a_i^T p \geq \cancel{\alpha +}\, \alpha\cancel{\beta} + \varepsilon \\ b_j^T p \leq \alpha \cancel{\beta} - \varepsilon \end{array}\right\} \text{ for some } \varepsilon > 0$$

Scale so that $\varepsilon = 1$, ie, take $\cancel{\beta}\, x = \frac{1}{\varepsilon} p$, $\gamma = \frac{\alpha}{\varepsilon}$:

Need
$$\begin{array}{l} a_i^T x - \gamma \geq 1 \\ -b_j^T x + \gamma \geq 1 \end{array}$$

Conversely: If the condition is satisfied then $x^T w = \gamma$ separates the two sets.

(b) Let $A = \begin{bmatrix} a_1^T \\ \vdots \\ a_k^T \end{bmatrix}$, $B = \begin{bmatrix} b_1^T \\ \vdots \\ b_m^T \end{bmatrix}$.

Part (a) can be written:
$$\begin{array}{ll} \min & 0^T x \\ \text{s.t.} & Ax - \gamma e \geq e \\ & -Bx + \gamma e \geq e \end{array}$$

Dual is:
$$\begin{array}{ll} \max & e^T y + e^T z \\ \text{st.} & A^T y - \cancel{B}^T z = 0 \\ & -e^T y + e^T z = 0 \\ & y, z \geq 0. \end{array}$$

If dual has a feasible nonzero solution then it has dual value >0, so the primal problem must be infeasible.

We can scale this feasible solution so that $e^T y = e^T z = 1$.

Then
$$0 = A^T y - B^T z = \underbrace{\sum a_i y_i}_{\text{Convex combination of points in } A} - \underbrace{\sum b_j z_j}_{\text{convex combination of points in } B.}$$

Thus, there is a point $(A^T y = B^T z)$ that is a convex combination of points in A and also a convex combination of points in B.

5. (a) $\min f(x)$

s.t. $x \geq 0$.

KKT gradient condition:

$$\nabla f(x) - u = 0$$

Complementarity: $u^T x = 0$

Nonnegativity: $u \geq 0$

Feasibility: $x \geq 0$.

Thus, ~~we~~ ~~a~~ eliminating u shows we need:

$$\nabla f(\bar{x}) \geq 0, \quad \bar{x} \geq 0, \quad \bar{x}^T \nabla f(\bar{x}) = 0.$$

(b) (i) Assume $\bar{x}$ and $\hat{x}$ are both zeroes of F.

But then $F(\bar{x}) - F(\hat{x}) = 0$, so $(F(\bar{x}) - F(\hat{x}))^T(\bar{x} - \hat{x}) = 0$,

so F is not strictly monotone.

(ii) Assume $\bar{x}, \hat{x}$ both solve the NLCP.

Then

$$(F(\bar{x}) - F(\hat{x}))^T(\bar{x} - \hat{x}) = F(\bar{x})^T\bar{x} + F(\hat{x})^T\hat{x} - F(\bar{x})^T\hat{x} - F(\hat{x})^T\bar{x}$$

$$\cancel{= F(} = -\hat{x}^T F(\bar{x}) - \bar{x}^T F(\hat{x}) \leq 0 \text{ since } \bar{x}, \hat{x}, F(\bar{x}), F(\hat{x}) \text{ all } \geq 0$$

So F is not strictly monotone.

(c) To be a minimizer over $\mathbb{R}^n$, we need to find a zero of ∇f. But at most one such zero exists, so f has at most one minimizer over $\mathbb{R}^n$.

To be a minimizer over $\mathbb{R}^n_+$, need to find a solution to the NLCP: $\bar{x} \geq 0$, $\nabla f(\bar{x}) \geq 0$, $\bar{x}^T \nabla f(\bar{x}) = 0$, from part (a).

From part (b), at most one such solution exists, if f is strictly convex.