PARAMETRIC EQUATIONS

Objective: The purpose of this lab is to study the effects of parameters on the shape and characteristics of curves, to demonstrate how Maple can be used to simplify calculations involving parametric equations, and gain an appreciation of the graphing capabilities of Maple.

Problem 1: If a circle C of radius B rolls on the inside of a second circle with radius A, a curve is traced out by a fixed point P on C. The curve can be described by the following set of equations:

\[ x := (A - B) \cos(\theta) + B \cos\left(\frac{(A - B) \theta}{B}\right) \]
\[ y := (A - B) \sin(\theta) - B \sin\left(\frac{(A - B) \theta}{B}\right) \]

Note: If A is larger than B, the curve is called a hypocycloid.

a) The special case in which \( B = \frac{A}{4} \) is called an astroid. Use Maple to plot the parametric equations for \( A = 4 \) and \( B = 1 \). Plot it for three different values of \( \theta \); \( \theta = 0 \ldots \pi \), \( 0.2 \pi \), and \( 0.4 \pi \). Only include the graph for \( \theta = 0.2 \pi \), but describe the results for the other two. Use the `plot([parametric])` command. Use Maple help, the Maple Study Guide or the Maple class session (on the course web page) for examples of its use.

b) Plot the parametric equations for \( A = 3 \), \( A = 5 \), and \( A = 8 \) while \( B = 1 \) and \( \theta = 0.2 \pi \). Only include the graph for \( A = 5 \) but describe your results for the other two.

c) Plot with \( B = 2 \), \( B = 3 \), and \( B = 5 \) while \( A = 1 \). Only include the graph for \( B = 3 \) but describe the results for the other two. Note that using \( \theta = 0.2 \pi \) does not trace out the entire curve. Increase the upper limit of \( \theta \) incrementally by \( 2 \pi \) to find the smallest value for which the entire curve is traced out. How is this value of \( \theta \) related to \( B \)?

d) Plot with different positive integer values of \( A \) and \( B \). Describe the curve when \( A > B \), except when \( A = 2B \). Why is \( A = 2B \) different? You may hand write or type the answers for this question.

e) Again plot the curve with different positive integer values of \( A \) and \( B \). Describe the curves when \( A < B \), when neither \( A \) nor \( B \) is equal to 1. What happens when \( A = B \) and why? (Do not include any plots.) You may hand write or type the answers for this question.

Extra Credit- (2 pts) - Describe the relationship between \( A \) and \( B \) that determines the number of cusps on the curve.
**Problem 2:** In this problem you will estimate the area of the region enclosed by the loop of the curve corresponding to the parametric equations:

\[
u := 1 + \tan(t) \left(1 - 2 \cos(t + 1)^2\right) \\
v := 1 - \frac{2}{3} \cos(t)^2\]

a) Plot the curve for the interval \([-\pi/3..\pi/3]\) with \(u\) on the horizontal axis and \(v\) on the vertical axis.

b) At what point \((u,v)\) does the curve cross itself? Estimate the \(u\) and \(v\) coordinates of the point of intersection by clicking the mouse on that point. Use the estimate of the value of \(v\) to find the values of \(t\) at the point of intersection. Call the smaller value \(t_0\) and the larger value \(t_2\). Check your answer by substituting \(t_0\) and \(t_2\) into the equation for \(u\).

c) Find the value of \(t\) in the interval \([-\pi/3..\pi/3]\) where the loop has a vertical tangent. Call this value \(t_1\).

d) At what value of \(t\) in the interval \([-\pi/3..\pi/3]\) does the loop have a horizontal tangent?

e) To determine the area of the loop in the interval \([-\pi/3..\pi/3]\), we divide the curve into 2 distinct sections. Section #1 is from the intersection to the vertical tangent, and Section #2 is from the vertical tangent to the intersection. Use the following command to see these two sections:

\[
\text{plot}([\{u,v, t=-0.33*\pi..t1\}, \{u,v, t=t1..0.33*\pi\}]);
\]

Include this plot in your worksheet. Label \(t_0\), \(t_1\), and \(t_2\) by hand, and draw an arrow from \(t_0\) to \(t_1\) and another arrow from \(t_1\) to \(t_2\).

f) Calculate the area under each section of the curve, and then calculate the total area \(A\) of the loop by adding the area under Section #1 and subtracting the area under Section #2. Use the following formula for \(A\):

\[
A := \int_{t_0}^{t_1} \sqrt{\left(\frac{du}{dt}\right)} \, dt - \int_{t_1}^{t_2} \sqrt{\left(\frac{du}{dt}\right)} \, dt
\]

Why do the limits on the first integral go from \(t_1\) to \(t_0\)?

g) Notice that in the equation for part (f), we can reverse the limits of integration for the second integral and add this new integral to the other integral. By doing this we obtain an equation that allows us to calculate the area by integrating \(v^*du\) from \(t_2\) to \(t_0\). Calculate this integral and verify that you obtain the same result as in part (e).