1. (I) has a solution

$\iff$ $Ax \leq -e$, $x \geq 0$ has a solution (e = vector of ones)

$\iff$ $Ax + s = -e$, $(x, s) \geq 0$ has a solution

$\iff$ $A^T y \geq 0$, $y \geq 0$, $-e^T y < 0$ has no solution by Farkas Lemma

$\iff$ $A^T y \geq 0$, $y \geq 0$, $y \neq 0$ has no solution

$\iff$ (II) does not have a solution

2. Let $A, B \in S^n_+$. Let $\bar{X} = \lambda A + (1-\lambda) B$, with $0 \leq \lambda \leq 1$.

Then $\lambda f(A) + (1-\lambda) f(B) - f(\lambda A + (1-\lambda) B)$

$$= \lambda \operatorname{tr}(CA^2) + (1-\lambda)\operatorname{tr}(CB^2)$$
$$- \operatorname{tr}\left(C(\lambda A + (1-\lambda)B)(\lambda A + (1-\lambda)B)\right)$$

$$= \operatorname{tr}\left(C\left[\begin{array}{l}\lambda A^2 + (1-\lambda)B^2 - \lambda^2 A^2 - (1-\lambda)^2 B^2 \\ \quad - \lambda(1-\lambda)AB - \lambda(1-\lambda)BA\end{array}\right]\right)$$

$$= \operatorname{tr}\left(C\left[\begin{array}{l}\lambda(1-\lambda)A^2 + \lambda(1-\lambda)B^2 \\ \quad - \lambda(1-\lambda)AB - \lambda(1-\lambda)BA\end{array}\right]\right)$$

$$= \lambda(1-\lambda)\operatorname{tr}\left(C(A-B)^2\right)$$

≥ 0 as required,

since $(A-B)(A-B)$ is ~~an~~ in S^n_+,

and the trace of the product of two matrices in S^n_+ (namely, C and $(A-B)^2$) is nonnegative.

3. (a) $\nabla f(x) = e^{x_2 - x_1}\begin{bmatrix} 2-2x_1-1 \\ 2x_1+1 \end{bmatrix} = e^{x_2-x_1}\begin{bmatrix} 1-2x_1 \\ 2x_1+1 \end{bmatrix}$

At KKT points, need:

$$e^{x_2-x_1}(1-2x_1) - u_1 = 0 \quad (1)$$
$$e^{x_2-x_1}(2x_1+1) - u_2 = 0 \quad (2)$$

$u_1 x_1 = 0$ (3) $u_1 \geq 0$ (4) $x_1 \geq 0$ (5)

$u_2 x_2 = 0$ (6) $u_2 \geq 0$ (7) $x_2 \geq 0$ (8)

(b)

Assume $\underline{u_2 = 0}$: $\overset{(2)}{\Longrightarrow} x_1 = -\frac{1}{2}$. Violates (5).

So must have $\boxed{u_2 > 0} \overset{(6)}{\Longrightarrow} \boxed{x_2 = 0}$

Assume $x_1 > 0$: $\overset{(3)}{\Longrightarrow} u_1 = 0 \overset{(1)}{\Longrightarrow} x_1 = \frac{1}{2} \overset{(2)}{\Longrightarrow} u_2 = 2e^{-\frac{1}{2}} > 0$.

So get KKT point: $\boxed{x = (\tfrac{1}{2}, 0) \text{ with } u = (0, 2e^{-\frac{1}{2}})}$.

Assume $x_1 = 0$: $\overset{(1)}{\Longrightarrow} u_1 = 1$, $\overset{(2)}{\Longrightarrow} u_2 = 1$

So get KKT point $\boxed{x = (0,0) \text{ with } u = (1,1)}$

(c) $\nabla^2 f = e^{x_2-x_1}\left\{\begin{bmatrix} 2x_1-1 & -2x_1+1 \\ -2x_1+1 & 2x_1+1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}\right\} = e^{x_2-x_1}\begin{bmatrix} 2x_1-3 & 1-2x_1 \\ 1-2x_1 & 2x_1+1 \end{bmatrix}$

Since $\nabla^2 g_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for both constraints, have $\nabla^2 L = \nabla^2 f$.

(i) $x = (0,0)$: Have $u_1 > 0$, $u_2 > 0$. So look at d with $d^T \nabla g_1 = d^T \nabla g_2 = 0 \Rightarrow d = (0,0)$.

So $d^T \nabla^2 L d > 0$ for all appropriate $d \neq (0,0)$, vacuously, so $\boxed{x = (0,0) \text{ is a local min.}}$

$x = (\frac{1}{2}, 0)$: Have $g_2(x) = 0$, so look at d with $d^T \nabla g_2 = 0$, so $d = (t, 0)$ for $t \neq 0$.

Then $d^T \nabla^2 L d = -2e^{-0.5} t^2 < 0$. So fail necessary cond. $\boxed{\text{So } x = (\tfrac{1}{2}, 0) \text{ not local min}}$

3(d) Consider the set of points $x = (t, 0)$.

As $t \to \infty$, get $f(x) \to 0$.

Now, $f(0,0) = 1$, and $f(\frac{1}{2}, 0) = 2e^{-1/2}$

So neither point is a local min.

(e) $$\nabla^2 f = e^{x_2 - x_1} \begin{bmatrix} 2x_1 - 3 & 1-2x_1 \\ 1-2x_1 & 2x_1+1 \end{bmatrix}$$

Determinant is

$$\Delta = e^{2(x_2-x_1)}\left[(2x_1-3)(2x_1+1) - (1-2x_1)^2\right]$$

$$= e^{2(x_2-x_1)}\left[4x_1^2 - 4x_1 - 3 - 1 + 4x_1 - 4x_1^2\right]$$

$$= -4e^{2(x_2-x_1)}$$

< 0 for every x

So f is not convex in any region.

4. $f(x) = \frac{1}{2} x^T Q x \qquad g(x) = \frac{1}{2} x^T x - \frac{1}{2}$

KKT: $\nabla f = Qx$, $\nabla g = I$.

Feasible region is convex, constraints are convex, and $x=0$ is strictly feasible. So Slater CQ holds, so suffices to look for KKT point (feasible region is compact).

So need $Qx + uIx = 0$ ①, $(\frac{1}{2} x^T x - \frac{1}{2}) u = 0$, $u \geq 0$, $g(x) \leq 0$

<u>If $u=0$</u>: Then $Qx = 0$, so either $x=0$, or x in nullspace of Q.

Then $f(x) = 0$.

<u>If $u>0$</u>: Then $Qx = -ux$, so u must be the negative of a negative eigenvalue of Q, and x is the corresponding eigenvector of norm one.

Then $f(x) = \frac{1}{2} x^T Q x = -\frac{1}{2} u x^T x = -\frac{1}{2} u$

So optimal choice is $u = -\lambda_{min}(Q)$, the negative of the most negative eigenvalue of Q, and x is corresponding eigenvector.

Optimal value: $\boxed{\frac{1}{2} \lambda_{min}(Q)}$ for primal.

(By assumption, $\lambda_{min}(Q) < 0$.)

4. Dual problem:

$$\theta(u) = \min_{x \in \mathbb{R}^n} \tfrac{1}{2} x^T Q x + \tfrac{1}{2} u x^T x - \tfrac{1}{2} u$$

$$= \min_{x \in \mathbb{R}^n} \tfrac{1}{2} x^T (Q + uI) x - \tfrac{1}{2} u$$

$$= \begin{cases} -\tfrac{1}{2} u & \text{if } Q + uI \text{ positive semidefinite} \\ -\infty & \text{otherwise} \end{cases}$$

So $\theta(u)$ is maximized by taking $u = -\lambda_{min}(Q)$. (>0)

Optimal dual value is $\tfrac{1}{2} \lambda_{min}(Q)$ = optimal primal value.